

Lecture 5: Quantum measurement

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Entanglement and measurement processes are what distinguishes the most quantum and classical physics. Although the formalism provides operational rules to describe a measurement, their interpretation are still a matter of debate, in particular about the role of the observer (see for instance Ref. [1]).

Measurement processes in quantum physics are associated with the probabilities and the idea of “collapse of the wave function”. Contrary to what happens in classical physics, a quantum measurement does not simply extract information about the state of the system. It forces the system to acquire the value obtained by the measurement. For example, if a particle was in a well-defined momentum state, its position is totally undetermined. But, by performing a position measurement, i.e. by asking the particle where it is, we obtain a well-defined value and we force the particle to acquire this position. Its momentum then becomes totally undetermined.

This is only one aspect of the problem. Measurement theory is associated with many subtle concepts, such as projection valued measurements (PVM), positive operator valued measurements (POVM), quantum non demolition (QND) measurements, weak measurement... We will discuss some of them in this chapter.

1 Projective measurements (PVM)

Let us start by describing standard measurements in quantum physics, namely projective measurements, as conceptualized by von Neumann. Technically, one speaks of projection-valued measurements (PVM), a term inherited from a general underlying mathematical theory.

1.1 Projective measurement of a pure state

In Lecture 1 we inferred the measurement principle from the following intuitive rule: repeating a measurement right after a first attempt should give the same outcome. We concluded that the possible results of a measurement are the eigenvalues o_j of the operator $\hat{O}$, associated with the orthogonal (normalized) eigenvectors $|\alpha_j^\nu\rangle$. The index j spans the set of distinct eigenvalues of $\hat{O}$ and ν accounts for possible degeneracies, see Fig. 1.

Single measurement. The result of an individual measurement is random. It yields o_j with a probability

$$P_j = |\hat{\mathcal{P}}_j|\psi\rangle|^2 \quad (\text{Born's rule}) \quad \text{where} \quad \hat{\mathcal{P}}_j = \sum_{\nu} |\alpha_j^\nu\rangle\langle\alpha_j^\nu| \quad (1)$$

is the projection operator onto the eigenspace of $\hat{O}$ associated with the eigenvalue o_j . After a measurement giving the value o_j , the state of the system is projected onto the corresponding

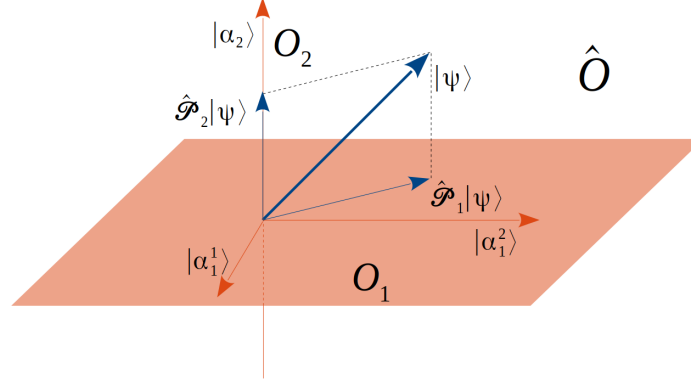


Figure 1: **Projective measurement.** Measuring an observable $\hat{O}$ on a system described by the ket $|\psi\rangle$, one obtains any of the eigenvalues o_j of $\hat{O}$. The probability to find the value o_j is the square modulus of the projection of the ket onto the corresponding eigenspace, $|\hat{\mathcal{P}}_j|\psi\rangle|^2$, generated by the kets $|\alpha_j^\nu\rangle$, where ν accounts for the possible degeneracy of o_j . If one obtains o_j , the ket is projected onto the corresponding eigenspace, and renormalized.

eigenspace and renormalized,

$$|\psi\rangle \rightarrow |\psi'_j\rangle = \frac{\hat{\mathcal{P}}_j|\psi\rangle}{|\hat{\mathcal{P}}_j|\psi\rangle|} \quad (\text{collapse principle}) . \quad (2)$$

This is the state of the system *conditioned* on the result of the measurement. A few remarks are in order.

1. The projection operators $\hat{\mathcal{P}}_j$ are sufficient to determine both the probability of a measurement result [Eq. (1)], and the after-measurement state of the system conditioned on the result of the measurement [Eq. (2)]. The result of a measurement is a projection operator, hence the name *projection-valued measurement* (PVM).
2. A PVM provides classical information in the sense that it determines the measurement result of a particular observable $\hat{O}$. It disregards the coherences between the different subspaces $\mathcal{E}_j(\hat{O})$ associated to the different measurement results.
3. The collapse principle and the fact that a projector fulfills $\hat{\mathcal{P}}_j^2 = \hat{\mathcal{P}}_j$ ensures that while the result of a first measurement is random, any subsequent measurement performed immediately after the first one on the same system returns the same result.
4. A single measurement does yield information about the system but it is important to understand which one. If the measurement gives o_j , we know that
 - (i) The initial state had a nonvanishing component in $\mathcal{E}_j(\hat{O})$. This is a very weak but non negligible amount of information (see below).
 - (ii) The final state is in $\mathcal{E}_j(\hat{O})$. If the value o_j is nondegenerate, the final state is exactly known (up to an irrelevant phase). This offers a way to prepare the system in a given state, although it requires a post selection conditioned to the result of the measurement.

Unread measurement. So far, we have assumed that the measurement was read. Suppose now that a measurement has been performed but its result is not read by the observer. This may seem like an academic question but we will see in Lecture 6 that unread measurements are useful to interpret the dynamics of open systems. In the case of an unread measurement, the state is projected onto one of the states $\hat{\mathcal{P}}_j|\psi\rangle/|\hat{\mathcal{P}}_j|\psi\rangle|$ with probability $P_j = |\hat{\mathcal{P}}_j|\psi\rangle|^2$. Considering the (classical) information we have, the after-measurement state is described by the density matrix

$$\hat{\rho}' = \sum_j P_j |\psi_{|j}\rangle\langle\psi_{|j}| = \sum_j P_j \frac{\hat{\mathcal{P}}_j|\psi\rangle\langle\psi|\hat{\mathcal{P}}_j}{|\hat{\mathcal{P}}_j|\psi\rangle|^2}, \quad (3)$$

that is

$$\hat{\rho}' = \sum_j \hat{\mathcal{P}}_j|\psi\rangle\langle\psi|\hat{\mathcal{P}}_j. \quad (4)$$

It can be checked that the after-measurement density matrix ρ' does fulfill all the conditions for being a density matrix, see exercise A. The fact that a measurement turns the description of the system state from a ket (pure state) to a density matrix (mixed state in general) illustrates the acquisition of classical information.

1.2 Projective measurement of a mixed state

Assume now that before the measurement we only have a partial knowledge of the system state. The initial state is thus not defined by a ket but by a density matrix:

$$\hat{\rho} = \sum_n \Pi_n |\psi_n\rangle\langle\psi_n|, \quad (5)$$

where the $|\psi_n\rangle$'s are possible states of the system, each with probability Π_n . Recall that the $|\psi_n\rangle$ are normalized but not necessarily mutually orthogonal.

Measurement probability. Let us perform a measurement of the observable $\hat{O}$. As before, one can obtain either of the values o_j with probability $P_j = \sum_n \Pi_n \times P_{j|n}$, which also reads (see derivation in exercise A)

$$P_j = \text{Tr}(\hat{\rho} \hat{\mathcal{P}}_j). \quad (6)$$

After-measurement state and Bayesian inference. Let us now determine the state of the system after the measurement, assuming that we have obtained the value o_j . Taking into account all the possible states of the system before the measurement, a naive calculation would yield

$$\tilde{\rho}'_{|j} = \sum_n \Pi_n \frac{\hat{\mathcal{P}}_j|\psi_n\rangle\langle\psi_n|\hat{\mathcal{P}}_j}{|\hat{\mathcal{P}}_j|\psi_n\rangle|^2}. \quad (7)$$

However, this expression is problematic as a state $|\psi_n\rangle$ may be orthogonal to the $\mathcal{E}_j(\hat{O})$, leading to $\hat{\mathcal{P}}_j|\psi_n\rangle = 0$: The quantity $\hat{\mathcal{P}}_j|\psi_n\rangle\langle\psi_n|\hat{\mathcal{P}}_j/|\hat{\mathcal{P}}_j|\psi_n\rangle|^2$ is then ill defined.

The above calculation ignores an important aspect of the measurement, namely that it provides information about the state of the system, which we did not necessarily have before the measurement. In particular, if we obtained the value o_j , we acquire the information that the system was not in a state orthogonal to $\mathcal{E}_j(\hat{O})$. This removes the ill-defined terms in $\tilde{\rho}'_{|j}$.

Indeed, we substitute the initial probability of the state $|n\rangle$ by the probability conditioned to the measurement result j , i.e.

$$\Pi_n \rightarrow P_{n|j} \quad (8)$$

This approach comes from classical probability theory, and is called *Bayesian inference*. Bayes' theorem tells us that

$$P_{n|j} = \frac{P_{j|n} \Pi_n}{P_j} . \quad (9)$$

Using the relation $P_{j|n} = |\hat{\mathcal{P}}_j |\psi_n\rangle|^2$, Eq. (6), and the substitution (8), we find

$$\hat{\rho}'_{|j} = \sum_n \frac{P_{j|n} \times \Pi_n}{P_j} \frac{\hat{\mathcal{P}}_j |\psi_n\rangle \langle \psi_n| \hat{\mathcal{P}}_j}{|\hat{\mathcal{P}}_j |\psi_n\rangle|^2} = \frac{\sum_n \Pi_n \hat{\mathcal{P}}_j |\psi_n\rangle \langle \psi_n| \hat{\mathcal{P}}_j}{\text{Tr}(\hat{\rho} \hat{\mathcal{P}}_j)} , \quad (10)$$

that is

$$\hat{\rho}'_{|j} = \frac{\hat{\mathcal{P}}_j \hat{\rho} \hat{\mathcal{P}}_j}{\text{Tr}(\hat{\rho} \hat{\mathcal{P}}_j)} . \quad (11)$$

If now the measurement is *not read*, we average the conditional results $\hat{\rho}'_{|j}$ over the measurement results. We find $\hat{\rho}' = \sum_j P_j \times \hat{\rho}'_{|j}$, which yields

$$\hat{\rho}' = \sum_j \hat{\mathcal{P}}_j \hat{\rho} \hat{\mathcal{P}}_j . \quad (12)$$

Again, $\hat{\rho}'_{|j}$ and $\hat{\rho}'$ fulfill all the properties of density matrices.

Quantum information approach. It is instructive to reconsider the above problem from a different point of view. Assume that the system A is coupled to another system B . If the bipartite system ($A \otimes B$) is isolated, the state may be written

$$|\Psi\rangle_{AB} = \sum_{n=1}^{d_B} \left[\sum_{m=1}^{d_A} a_{n,m} |\phi_m\rangle_A \right] \otimes |\chi_n\rangle_B = \sum_{n=1}^{d_B} c_n |\psi_n\rangle_A \otimes |\chi_n\rangle_B , \quad (13)$$

where $|\phi_m\rangle_A, |\chi_n\rangle_B$'s form an orthonormal basis of $\mathcal{E}_{A,B}$, and the $|\psi_n\rangle_A$'s form a family of normalized states of $\mathcal{E}_A$ (not necessarily orthogonal). The normalization of the bipartite state $|\Psi\rangle_{AB}$ implies

$$\sum_{n=1}^{d_B} |c_n|^2 = 1 . \quad (14)$$

The associated reduced density matrix of the sub-system A then reads

$$\hat{\rho}_A = \sum_n |c_n|^2 |\psi_n\rangle \langle \psi_n| . \quad (15)$$

Let us now perform a measurement on the subsystem A only, corresponding to the observable $\hat{O}_A \otimes \hat{I}_B$. Similarly, the projectors associated to the measurement are $\hat{\mathcal{P}}_{j,A} \otimes \hat{I}_B$. From the bipartite state (13) we get the probability of obtaining o_j

$$P_j = |(\hat{\mathcal{P}}_{j,A} \otimes \hat{I}_B) |\Psi\rangle|^2 = \text{Tr}(\hat{\mathcal{P}}_j \hat{\rho}_A) , \quad (16)$$

as demonstrated in exercise A. Assume that we have obtained the result o_j . The bipartite state conditioned on this measurement result is [see Eq. (2)]

$$|\Psi'_{|j}\rangle = \frac{(\hat{\mathcal{P}}_{j,A} \otimes \hat{I}_B) \sum_n c_n |\psi_n\rangle_A \otimes |\chi_n\rangle_B}{\sqrt{P_j}} = \frac{\sum_n c_n (\hat{\mathcal{P}}_j |\psi_n\rangle)_A \otimes |\chi_n\rangle_B}{\sqrt{P_j}}. \quad (17)$$

It corresponds to the reduced density matrix

$$\hat{\rho}'_{A|j} = \frac{\sum_n |c_n|^2 \hat{\mathcal{P}}_j |\psi_n\rangle\langle\psi_n| \hat{\mathcal{P}}_j}{P_j}, \quad (18)$$

which is nothing but Eq. (11). We thus find by the quantum information approach the same results as obtained by the density matrix approach. Here, we did not need to explicitly use Bayesian inference, i.e; we did not need to modify by hand the probabilities of the various possible states of A . The reason is that the projection is performed on the bipartite space $\mathcal{E}_A \otimes \mathcal{E}_B$ and it is the ket in this space that is globally renormalized. In the subspace of $\mathcal{E}_A$, this corresponds to projecting the reduced density matrix, $\hat{\rho} \rightarrow \hat{\mathcal{P}}_j \hat{\rho} \hat{\mathcal{P}}_j$, and renormalize by $\text{Tr}(\hat{\mathcal{P}}_j \hat{\rho} \hat{\mathcal{P}}_j) = \text{Tr}(\hat{\rho} \hat{\mathcal{P}}_j)$, which gives Eq. (11).

2 Quantum non-demolition measurements

We now study a class of measurements specific to quantum mechanics, called *quantum non-demolition measurements* (QND measurements).

2.1 Motivation

Usual measurements in quantum physics are *destructive*. This contains two ideas that should not be mixed up.

First, the state of the system is modified by the measurement. There is nothing we can do against it. To the best of our knowledge, the collapse rule is a fundamental aspect of quantum physics and, up to now, no experience has ever challenged it. In classical physics instead, the coupling to a measurement device may affect the system's dynamics but it is a perturbation, which can, in principle, be arbitrarily reduced. In contrast, a PVM in quantum physics dramatically affects the system state, whatever the coupling strength to the measurement device.

Second, the physical integrity of the system may be affected by the measurement process. In extreme cases, it does not even exist any longer after the measurement. The system is thus *demolished*. In fact, standard quantum measurements are demolition measurements. This is the case in the Stern and Gerlach experiment, the Young slit experiment with photons or electrons, photodetection of single or multi-photon states, as well as time-of-flight imaging in ultracold atoms, to name a few. However, in contrast to wavefunction collapse, this is a practical rather than fundamental issue and, one can in principle overcome it. In the following, we will describe a general approach to realize a full QND measurement using quantum entanglement.

Before proceeding, let us stress that standard (demolition) measurements have many disadvantages. Firstly, as mentioned above, an important property of PVMs in quantum physics is that, although the result of a measurement is random, two subsequent measurements of

the same quantity always yield the same result ($\hat{\mathcal{P}}_j^2 = \hat{\mathcal{P}}_j$). However, this property cannot be tested experimentally if the system is destroyed after the first measurement. Secondly, as already mentioned above, an efficient method to prepare a system in a given state is to measure it in this state: Assume you have a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and measure it in the basis $\{|0\rangle, |1\rangle\}$; If the measurement returns 0, the qubit is now for sure in $|0\rangle$; If the measurement returns 1, the qubit is for sure in $|1\rangle$. However, if the system is destroyed by the measurement, it is of little use for subsequent quantum information processing! Finally, it is not possible to track the time evolution of a quantum system. This is most often not a serious issue as a measurement, whatever it is, would alter the unitary evolution of the system. In contrast, it is a serious issue to demonstrate effects induced by repeated measurements such as the Zeno effect.

2.2 QND measurement: Formal definition

Let us first give the formal definition: *a QND measurement is a measurement that can be repeated a large number of times, at arbitrary times, and always returns the same result.* This definition has two immediate consequences. First, the system must be available after the measurement. Second, it requires that the evolution of the system between two measurements maintains it in the eigenspace corresponding to the result of the first measurement, as illustrated on Fig 2. Thus, a quantum measure is QND if and only if

- (i) The system is available after the measurement;
- (ii) The observable $\hat{O}$ commutes with the evolution operator $\hat{U}(t)$ at any time, or equivalently, with the Hamiltonian $\hat{H}(t)$ at any time.

Let us give examples of QND or non-QND measurements, assuming that the system is available after the measurement. For a free particle, governed by the Hamiltonian $\hat{H} = \hat{p}^2/2m$, $\hat{p}$ is a QND observable but $\hat{r}$ is not. For a harmonic oscillator, $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$, neither $\hat{x}$ or $\hat{p}$ is QND but the excitation number $\hat{n} = \hat{a}^\dagger\hat{a}$ is QND. For a $1/2$ -spin in a magnetic field $\mathbf{B} = B\mathbf{e}_x$, $\hat{S}_x$ is QND but $\hat{S}_y$ and $\hat{S}_z$ are not. For any isolated system, $\hat{H}$ is QND.

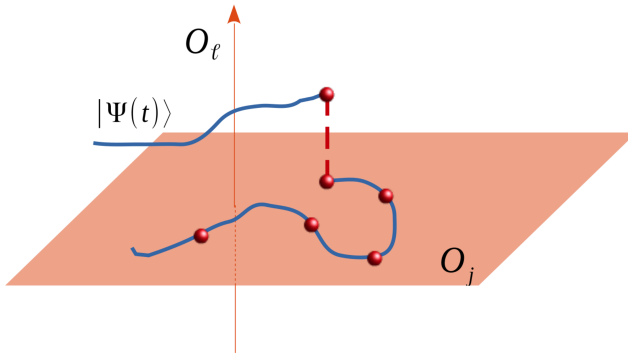


Figure 2: Sketch of a QND measurement, according to the definition of Sec. 2.2. The first measurement projects the state on the eigenspace of the observable corresponding to the obtained result. The subsequent evolution – unitary evolution (solid blue line) interrupted by measurements (red points) – maintains the state in the same eigenspace.

2.3 QND measurement in practice

From now on, we adopt the more “relaxed” definition of QND measurements requiring only the condition (i) to hold, namely that the system is available after the measurement. The idea to perform a QND measurement is to couple the system of interest to an auxilliary system and perform a standard, i.e. demolishing, measurement of this auxilliary system, without “touching” the system of interest. Of course, the aim is always to carry out a true measurement of the system state, i.e. to know in which state it is (after the measurement).

General principle. The principle of a QND measurement is the following: Assume that the system S you want to measure is in the superposition state $|\psi\rangle = \sum_n c_n |n\rangle$, where $\{|n\rangle\}$ is an orthonormal basis. Then prepare an ancilla A into some reference state $|0\rangle$ and couple it to the system using an appropriate evolution operator $\hat{U}$, so as to create an entangled state of the form

$$|\Psi\rangle = \sum_n c_n |n\rangle \otimes |\chi_n\rangle, \quad (19)$$

where $\{|\chi_n\rangle\}$ forms an orthonormal family (this requires that the dimension of the ancilla Hilbert space is at least equal to that of the system of interest). Then, assume you can perform a standard PVM in the $\{|\chi_n\rangle\}$ family by coupling the ancilla to a measurement apparatus, without touching the system. The probability to find $|\chi_n\rangle$ is $P_n = |\hat{\mathcal{P}}_n |\Psi\rangle|^2 = |c_n|^2$, where $\hat{\mathcal{P}}_n = \hat{I}_S \otimes \hat{\mathcal{P}}_{A,n}$ is the projector onto the ancilla Hilbert space associated to $|\chi_n\rangle$. This probability is equal to that of the system being initially in the state $|n\rangle$. Moreover, assuming you find $|\chi_n\rangle$, the bipartite state is projected onto the conditional pure state

$$|\Psi\rangle \rightarrow |\Psi'_n\rangle = \frac{\hat{\mathcal{P}}_n |\Psi\rangle}{|\hat{\mathcal{P}}_n |\Psi\rangle|} = |n\rangle \otimes |\chi_n\rangle. \quad (20)$$

After the measurement, the system and the ancilla are no longer entangled and the system is in the pure state $|n\rangle$. More precisely, in many cases, the state $|n\rangle_S \otimes |\chi_n\rangle_A$ is actually improper because the ancilla is physically destroyed by the PVM, and we should rather restrict the after-measurement state to that of the system, i.e. $|n\rangle$. Neither the coupling $\hat{U}$ nor the PVM on A has destroyed S , which is still available for subsequent processes. In the end, we have thus realized an effective PVM on S without destroying it, i.e. a QND measurement of S .

Example. Assume that the system S is a qubit. To perform a QND measurement, we need an ancilla that lives in a Hilbert space at least as large, so at least another qubit. For simplicity, we shall use here a 1/2-spin representation. Assume that the system of interest S is in a state

$$|\psi\rangle_S = \alpha |+\rangle_S + \beta |-\rangle_S. \quad (21)$$

The ancilla is prepared, for example, in the state

$$|+y\rangle_A = \frac{1}{\sqrt{2}} (|+\rangle_A + i |-\rangle_A) \quad (22)$$

and we apply the Ising-type coupling Hamiltonian

$$\hat{H} = \frac{\hbar g}{2} \hat{\sigma}_S^z \otimes \hat{\sigma}_A^z. \quad (23)$$

For the discussion, we take an antiferromagnetic coupling $g > 0$. Since $\hat{H}$ is diagonal in the computational basis $\{|\pm z\rangle_S \otimes |\pm z\rangle_A\}$, its action is the following:

- If S is in the state $|+z\rangle$, $\hat{H}$ acts on A as a magnetic field along the z axis. Since $g > 0$, it induces a counter-clockwise precession of the qubit A around z . The interaction strength and times are tuned so as to realize a $\pi/2$ pulse, i.e. $t = \pi/2g$. It realizes the transformation:

$$|+y\rangle_A \longrightarrow \frac{1}{\sqrt{i}} |-x\rangle_A . \quad (24)$$

- If S is in the state $|-z\rangle$ the effective magnetic field acting on the ancilla A is inverted and the rotation on the Bloch sphere is now clockwise, and

$$|+y\rangle_A \longrightarrow \sqrt{i} |+x\rangle_A . \quad (25)$$

Since the coupling is unitary, at the end of the interaction process, the bipartite system $S \otimes A$ ends up in the entangled state

$$|\Psi_{SA}\rangle = \frac{\alpha}{\sqrt{i}} |+z\rangle_S \otimes |-x\rangle_A + \sqrt{i}\beta |-z\rangle_S \otimes |+x\rangle_A . \quad (26)$$

We now perform a PVM of the spin of A along the x axis. There are two cases to consider:

- We find the ancilla in $|-x\rangle_A$ with probability $|\alpha|^2$ and the system S is then projected onto $|+z\rangle_S$;
- We find the ancilla in $|+x\rangle_A$ with probability $|\beta|^2$ and the system S is then projected onto $|-z\rangle_S$.

We have hence realized a QND measurement of the spin state of S .

2.4 QND measurement in cavity quantum electrodynamics

We now discuss an implementation of a QND measurement in a cavity quantum electrodynamics (CQED) experiment. The goal is to measure the number of photons in a cavity C , without destroying the photons, using Rydberg atoms. The system is the photon field in the cavity mode and the ancilla is a Rydberg atom. In the following, we assume that the cavity mode has a frequency ω and contains at most one photon. The most general state is thus

$$|\psi\rangle_C = \alpha |0\rangle + \beta |1\rangle . \quad (27)$$

The Rydberg atoms are considered as two-level systems, with ground state $|g\rangle$, excited state $|e\rangle$, and atomic angular frequency ω_A .

The experimental apparatus is shown in Fig. 3. The Rydberg atoms are emitted from the box B , prepared in the excited state $|e\rangle$ and injected one by one in the apparatus. Each atom is then turned into an equal superposition of the ground and excited states by a $\pi/2$ microwave pulse in the cavity R_1 . The latter contains a resonant quasi-classical field, at the frequency ω_A . The atom undergoes a Rabi oscillation of unit amplitude and period $2\pi/\Omega_R$ where Ω_R is

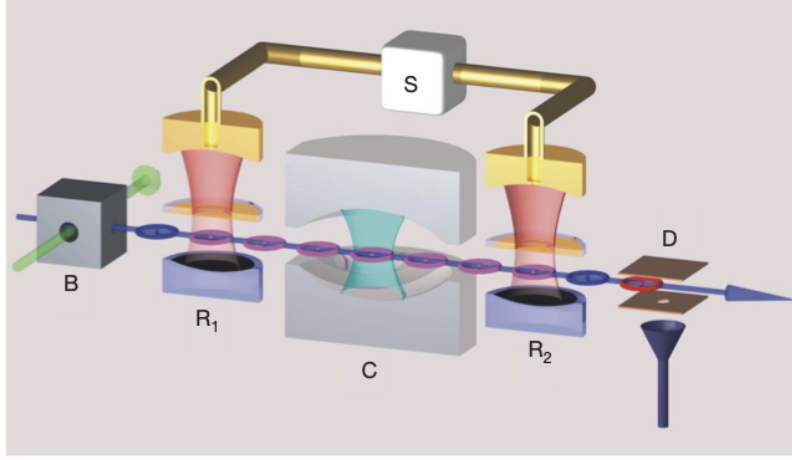


Figure 3: *Experimental setup used for the quantum non-demolition measurement of the photon state in a cavity. Figure from [3].*

the Rabi frequency. By setting the interaction time to $T_R = \pi/2\Omega_R$, the atom ends up into the state

$$|\chi'\rangle_A = \frac{1}{\sqrt{2}}(-i|g\rangle + |e\rangle) . \quad (28)$$

To simplify the notations, here and in the following, we systematically ignore all the dynamical phases such as $e^{-i\omega_A T_R}$. This transformation corresponds to a rotation of the Bloch vector around the x axis from $|e\rangle = |-z\rangle$ to $-i|+y\rangle$, in spin notations.

The atom then enters the cavity C and interacts with the cavity field to be measured, prepared in the state (27). The cavity mode frequency ω is detuned from the atomic resonance by $\delta = \omega - \omega_A < 0$. This situation is described by the Jaynes-Cummings Hamiltonian studied in Lecture 2:

$$\hat{H}_{JC} = \hbar\omega_A |e\rangle\langle e| + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\hbar\Omega}{2} (|e\rangle\langle g|\hat{a} + |g\rangle\langle e|\hat{a}^\dagger) .$$

In the weak coupling regime considered here, $|\delta| \gg \Omega\sqrt{n+1}$, the eigenstates $|g, 0\rangle$, $|g, n+1\rangle$, and $|e, n\rangle$ are nearly unperturbed. The ground state $|g, 0\rangle$ is isolated and has energy $E_{g,0} = 0$. The energies of all other states are calculated using second-order perturbation theory:

$$E_{g,n+1} \simeq E_{g,n+1}^0 - \frac{\hbar\Omega^2}{4|\delta|}(n+1) \quad \text{and} \quad E_{e,n} \simeq E_{e,n}^0 + \frac{\hbar\Omega^2}{4|\delta|}(n+1) , \quad (29)$$

with $E_{g,n+1}^0 = \hbar\omega(n+1)$ and $E_{e,n}^0 = \hbar\omega_A + \hbar\omega n$ the unperturbed energies. Now, the state of the atom-field system at the input of cavity C is

$$\begin{aligned} |\Psi'\rangle_{AC} &= \frac{1}{\sqrt{2}}(-i|g\rangle + |e\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= \frac{\alpha}{\sqrt{2}}(-i|g, 0\rangle + |e, 0\rangle) + \frac{\beta}{\sqrt{2}}(-i|g, 1\rangle + |e, 1\rangle) . \end{aligned} \quad (30)$$

After an interaction time T_C between atom and cavity, we obtain

$$\begin{aligned} |\Psi''\rangle_{AC} = & \frac{\alpha}{\sqrt{2}} \left(-i |g, 0\rangle + e^{-i\omega_A T_C} e^{-i\phi} |e, 0\rangle \right) \\ & + \frac{\beta}{\sqrt{2}} e^{-i\omega T_C} \left(-i e^{i\phi} |g, 1\rangle + e^{-i\omega_A T_C} e^{-i2\phi} |e, 1\rangle \right), \end{aligned} \quad (31)$$

with $\phi = \Omega^2 T_C / 4 |\delta|$. By setting $\phi = \pi/2$ and assuming $\omega T_C \equiv \omega_A T_C \equiv 0 \pmod{2\pi}$ to simplify the calculations, we get

$$|\Psi''\rangle_{AC} = -\frac{i\alpha}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |0\rangle + \frac{\beta}{\sqrt{2}}(|g\rangle - |e\rangle) \otimes |1\rangle. \quad (32)$$

The interaction has thus generated an atom-field entangled state. The zero and one photon states, $|0\rangle$ and $|1\rangle$, are coupled to the atomic states $(|g\rangle + |e\rangle)/\sqrt{2} = |+x\rangle$ and $(|g\rangle - |e\rangle)/\sqrt{2} = |-x\rangle$. By performing a PVM of the atomic state in this basis, we realize in principle a QND measurement of the photonic state. The probability amplitudes have been preserved, up to a phase, by the coupling, so that we obtain $|0\rangle$ and $|1\rangle$ with the probabilities $|\alpha|^2$ and $|\beta|^2$ of the initial cavity state.

It is, however, not straightforward to measure the atom in the appropriate basis. We know how to measure in the $\{|g\rangle, |e\rangle\}$ basis. We may for instance subject the atom to a field resonant with a $|e\rangle \rightarrow |2\rangle$ transition, but off resonant with any transition from $|g\rangle$, and perform a π pulse. If a photon is absorbed, the atom was in $|e\rangle$. Otherwise it was in $|g\rangle$. Such an approach cannot be extended to measure in $|\pm x\rangle$ because the two states have the same energy, at least without forbidding the transition from one of the states by some symmetries. To circumvent this difficulty, we rotate the atomic state, independently of the photonic state, before measuring it. This is realized by the cavity R_2 . It contains a resonant quasi-classical field, similar to that in cavity R_1 but with a relative phase $-\pi/2$. It realizes a rotation of the atomic state by $\pi/2$ around the y axis in a counter-clockwise direction,

$$\frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \longrightarrow |g\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}}(|g\rangle - |e\rangle) \longrightarrow -|e\rangle, \quad (33)$$

and the atom-cavity state is now

$$|\Psi'''\rangle_{AC} = -i\alpha |g\rangle \otimes |0\rangle - \beta |e\rangle \otimes |1\rangle. \quad (34)$$

Measuring the atomic state in the standard basis $\{|g\rangle, |e\rangle\}$, the atom is measured in $|g\rangle$ with probability $|\alpha|^2$, and the cavity field is projected onto $|0\rangle$; If the atom is measured in $|e\rangle$ (probability $|\beta|^2$), the cavity field is projected onto $|1\rangle$. We have thus realized a QND measurement of the photon number in the cavity mode. This scheme has been applied to a variety of QND measurements, including the determination of photon numbers [2] and well as the observation of the birth and death of a thermal photon in the cavity [3].

3 Generalized measurements (POVM)

It turns out however that many situations associated with real measurements are not adequately described by the PVMs formalism. Let us illustrate this on several examples.

Consider first the QND measurements in CQED discussed in the previous section: if we do not apply the last rotation of the atomic state as realized by the cavity R_2 , the atom cavity state is given by Eq. (32), which can also be written:

$$|\Psi''\rangle_{AC} = \frac{1}{\sqrt{2}}(-i\alpha|0\rangle + \beta|1\rangle) \otimes |g\rangle + \frac{1}{\sqrt{2}}(-i\alpha|0\rangle - \beta|1\rangle) \otimes |e\rangle . \quad (35)$$

Hence, the measurement of the atomic state $|g\rangle$ or $|e\rangle$ projects the cavity state onto $-i\alpha|0\rangle \pm \beta|1\rangle$. In general, these states are *not* orthogonal and therefore cannot be associated to the orthogonal eigenspaces of a Hermitian observable $\hat{O}$ acting in the Hilbert space of the cavity C . This means that the measurement process cannot be described by a projector acting on the system, as for this to hold, the outcome states have to be orthogonal.

Consider next the situation where a detector has a finite detection efficiency, as is always the case experimentally. Suppose for example that the detector can measure a qubit state $|0\rangle$ with a probability $1 - \epsilon$. This means that it will register a qubit actually prepared in $|0\rangle$ as a qubit $|1\rangle$ with a probability ϵ . Such a situation cannot be described by a projector, as repeating the measurement following a first one that gave an outcome $|0\rangle$, we may get $|1\rangle$. The property $\hat{\mathcal{P}}_0^2 = \hat{\mathcal{P}}_0$ is thus not fulfilled.

As another example, the detection of the state of an atom can be realized by ionization: the atom is subjected to an electric field gradient. The ionization starts at the point where the energy transfer equals the ionization energy. Since this ionization energy depends on the atomic state, we can deduce the latter from the position where the ionization started. The operators describing the action of the measurement read

$$\hat{M}_g = |\text{ion}\rangle\langle g| \quad \text{and} \quad \hat{M}_e = |\text{ion}\rangle\langle e| , \quad (36)$$

which are not projectors. In particular, $\hat{M}_g^2 = \hat{M}_e^2 = 0$, and the measurement cannot be repeated.

As a final example, consider a photon counter in a given mode of the radiation field. The absorption of a photon extracts an electron from the detector, which may then be amplified and measured. The resulting electric current is proportional to the number of absorbed photons and can therefore be determined. The photons are all absorbed, a process described by the operators

$$\hat{M}_n = |0\rangle\langle n| . \quad (37)$$

Again, these operators are not projectors in the Hilbert space of the radiation mode. In particular, $\hat{M}_n^2 = \delta_{n,0} |0\rangle\langle 0|$.

All these examples indicate that PVMs *à la von Neumann* are only a particular example of a more general class of measurements in quantum physics. Their framework is often too restrictive and not adapted to many real experimental situations. We thus need to develop a more general framework to describe quantum measurements.

3.1 Describing a generalized measurement from the unitary evolution of a larger system

In order to describe the most general quantum measurement processes on a system S , we have to include the measuring apparatus M , often called the *meter*. The meter, which does not need

to be explicitly described, is either a standard macroscopic measuring device, or in the case of QND measurements, the same device coupled to the ancilla A .

The system S is prepared in a state $|\psi\rangle_S$ and the meter in a reference state $|0\rangle_M$,

$$|\Psi\rangle_{SM} = |\psi\rangle_S \otimes |0\rangle_M . \quad (38)$$

When the system and the meter interact, entanglement is generally created between the two, and the state evolves under a unitary operator:

$$|\Psi'\rangle_{SM} = \hat{U} |\Psi\rangle_{SM} = \sum_m |m\rangle_M {}_M\langle m| \hat{U} |0\rangle_M |\psi\rangle_S = \sum_m \left(\hat{M}_m |\psi\rangle_S \right) \otimes |m\rangle_M , \quad (39)$$

where the states $|m\rangle_M$ describe an orthonormal basis of the meter's Hilbert space and where we have introduced the so-called *Kraus operators*

$$\hat{M}_m = {}_M\langle m| \hat{U} |0\rangle_M . \quad (40)$$

acting on the system only, i.e $\hat{M}_m$ is an operator on $\mathcal{E}_S$.

The states $|m\rangle_M$ represent the set of possible results of a measurement read on the meter while the Kraus operators $\hat{M}_m$ describe the action of the measurement process on the system S *conditioned* to the measurement result read on M . In the case of a von Neumann measurement described by an observable $\hat{O}$ on $\mathcal{E}_S$, these operators are the projectors $\hat{P}_j$ on the eigensubspaces of $\hat{O}$. Here, the operators $\hat{M}_m$ are in general not projectors, as we discussed above. Besides, they are not necessarily hermitian (see exercise B). The number of such operators is equal to the dimension of the Hilbert space of the meter, $\dim(\mathcal{E}_M)$, possibly larger than the dimension of the Hilbert space of the system, $\dim(\mathcal{E}_S)$.

The operators $\hat{M}_m$ are not arbitrary. Indeed, the coupling between the system and the meter is described by a unitary evolution operator $\hat{U}$, thus preserving the norm of the bipartite state $|\Psi\rangle_{SM}$. It follows the relation

$$1 = \langle \Psi' | \Psi' \rangle = \sum_m {}_S\langle \psi | \hat{M}_m^\dagger \hat{M}_m | \psi \rangle_S ,$$

since the states $|m\rangle_M$ form an orthonormal family. This relation being valid for any state of the system S , it implies the *completeness relation*

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}_S . \quad (41)$$

This identity generalizes the completeness relation on a set of orthogonal projectors, $\sum_j \hat{P}_j = \sum_j \hat{P}_j^2 = \hat{I}_S$. More precisely, as shown below, the Kraus operators $\hat{M}_m$ are the counterparts of the projection operators $\hat{P}_j$ in generalized measurement theory. Generalized quantum measurements are completely determined by the set of Kraus operators $\hat{M}_m$, just as the set of projectors $\hat{P}_j$ determine PVMs. Generalized measurements are called *positive operator valued measurements (POVM)*. The Kraus operators $\hat{M}_m$, however, need not be positive operators. Here the term “positive” refers to the quantities $\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$, which are indeed positive operators and are sufficient to determine the measurement probabilities, as we show below.

A POVM is thus *the complete set of operators* $\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$ associated to a measurement process, fulfilling $\sum_m \hat{E}_m = \hat{I}_S$.

As an example relevant in quantum optics, the annihilation operator $\hat{a}$ can be seen as the Kraus operator associated to the detection of a photon. Indeed, if we consider a monomode field containing n photons and a detector initially in its ground state $|0\rangle_D$, the interaction between the field and the detector leading to the absorption of one photon during a time dt is described by the unitary evolution:

$$|n\rangle \otimes |0\rangle_D \rightarrow \hat{U} |n\rangle \otimes |0\rangle_D = \sqrt{\eta\phi dt} |n-1\rangle \otimes |\text{click}\rangle_D + \sqrt{1-\eta\phi dt} |n\rangle \otimes |0\rangle_0, \quad (42)$$

with η the detection efficiency and ϕ the photon flux arriving on the detector. We have also assumed $\phi dt \ll 1$. Hence the Kraus operator $\hat{M}_1$ describing the detection of one photon fulfills $\hat{M}_1|n\rangle = \sqrt{\eta\phi dt} |n-1\rangle$, so that $\hat{M}_1 \propto \hat{a}$.

3.2 Generalized measurement of a pure state

Assume that the result of the measurement on the meter is read. This process is described by a PVM performed on M . Conditioned to the measurement result, the state of the meter is projected onto one of the states $|m\rangle$. According to the collapse principle applied to $S \otimes M$, the state after the measurement is given by the renormalized projection on $|m\rangle_M$, described by $\hat{\mathcal{P}}_m = \hat{I}_S \otimes |m\rangle\langle m|_M$, i.e.

$$|\Psi'_{|m}\rangle = \frac{\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M}{|\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M|}. \quad (43)$$

This state being a product state of S and M , the states of both subsystems are well defined and

$$|\psi'_{|m}\rangle_S = \frac{\hat{M}_m|\psi\rangle_S}{|\hat{M}_m|\psi\rangle_S|.} \quad (44)$$

This is the state of S conditioned to the result of the measurement performed on M .

According to Born's rule, the probability of measuring m is

$$P_m = |\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M|^2 = |\hat{M}_m|\psi\rangle_S|^2 = {}_S\langle\psi|\hat{E}_m|\psi\rangle_S. \quad (45)$$

It is crucial that the states $|m\rangle_M$ form an orthonormal set. This allows us to consider them as the results of a standard PVM on the meter, as well as for Eq. (44) to hold.

Now, if the result of the measurement is unread, the after-measurement state of S is described by the density matrix

$$\hat{\rho}' = \sum_m P_m |\psi'_{|m}\rangle\langle\psi'_{|m}| = \sum_m P_m \frac{\hat{M}_m|\psi\rangle\langle\psi|\hat{M}_m^\dagger}{|\hat{M}_m|\psi\rangle|^2}, \quad (46)$$

that is

$$\hat{\rho}' = \sum_m \hat{M}_m |\psi\rangle\langle\psi| \hat{M}_m^\dagger. \quad (47)$$

These results directly generalize those of Sec. 1.1 by replacing the projection operators $\hat{\mathcal{P}}_j$ by the Kraus operators $\hat{M}_m$ or $\hat{M}_m^\dagger$ depending on whether they are associated to a ket or a

bra. The same result can be obtained by computing the partial trace on the meter states of the bipartite density matrix after interaction, $|\Psi'\rangle\langle\Psi'|$.

As an example, consider the action of a weak beamsplitter on a monomode light field in a state $|\psi\rangle$. A detector is placed in the path weakly reflected by the beamsplitter. A clic on the detector prepares conditionally the transmitted field state:

$$|\psi'\rangle = \frac{\hat{a}|\psi\rangle}{\sqrt{\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle}}. \quad (48)$$

This method is used in practice to generate non-classical states of light.

3.3 Generalized measurement of a mixed state

Let us now generalize the notion of POVMs to the case where the system S is in a mixed state, described by the density matrix

$$\hat{\rho} = \sum_n \Pi_n |\psi_n\rangle\langle\psi_n|. \quad (49)$$

We will here use the *purification approach* introduced in Sec. 1.2. It is easier than the direct density-matrix approach because the Bayesian inference is built in, as we discussed in that Section. To describe the mixed state of S , we assume that the system is in an entangled state with some environment E ,

$$|\Psi\rangle_{SE} = \sum_n c_n |\psi_n\rangle_S \otimes |\chi_n\rangle_E, \quad (50)$$

where the coefficients c_n are such that $|c_n|^2 = \Pi_n$, with any phase, and the $|\chi_n\rangle$'s form an orthonormal family of $\mathcal{E}_E$. This approach is valid as long as we only consider measurements performed on S and not on E , since no measurement on S can distinguish a mixed state from a state that is entangled with an unobserved system. In practice, E can be a real or fictitious environment of the system. The tripartite state before the measurement reads

$$|\Psi\rangle_{SEM} = \sum_n c_n |\psi_n\rangle_S \otimes |\chi_n\rangle_E \otimes |0\rangle_M. \quad (51)$$

We then perform a measurement on S alone by coupling it to the measuring device. The tripartite state resulting from the coupling becomes

$$|\Psi'\rangle_{SEM} = \sum_m \left[\sum_n c_n \left(\hat{M}_m |\psi_n\rangle_S \right) \otimes |\chi_n\rangle_E \right] \otimes |m\rangle_M. \quad (52)$$

We have assumed here that the measuring device only acts on S , so that the interaction does not affect the environment E . Nevertheless, the entanglement created between S and M induces an entanglement between E and M . The state of the system $S \otimes E$ conditioned to the result of the measurement performed on M is

$$|\Psi'_{|m}\rangle_{SE} = \frac{\sum_n c_n \left(\hat{M}_m |\psi_n\rangle_S \right) \otimes |\chi_n\rangle_E}{\left| \sum_n c_n \left(\hat{M}_m |\psi_n\rangle_S \right) \otimes |\chi_n\rangle_E \right|}. \quad (53)$$

Tracing over the environment degrees of freedom, we obtain the density matrix of S

$$\hat{\rho}'_m = \frac{\sum_n |c_n|^2 \hat{M}_m |\psi_n\rangle\langle\psi_n| \hat{M}_m^\dagger}{\sum_n |c_n|^2 \langle\psi_n| \hat{M}_m^\dagger \hat{M}_m |\psi_n\rangle} , \quad (54)$$

that is

$$\hat{\rho}'_m = \frac{\hat{M}_m \hat{\rho} \hat{M}_m^\dagger}{\text{Tr}(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger)} . \quad (55)$$

The probability of measuring m is

$$P_m = \left| \sum_n c_n \left(\hat{M}_m |\psi_n\rangle_S \right) \otimes |\chi_n\rangle_E \right|^2 = \sum_n |c_n|^2_S \langle\psi_n| \hat{M}_m^\dagger \hat{M}_m |\psi_n\rangle_S . \quad (56)$$

It yields

$$P_m = \text{Tr}(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger) = \text{Tr}(\hat{E}_m \hat{\rho}) , \quad (57)$$

where $\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$.

Finally, if the measurement is unread, we find $\hat{\rho}' = \sum_m P_m \times \hat{\rho}'_m$. Combining Eqs. (55) and (57), it leads to

$$\hat{\rho}' = \sum_m \hat{M}_m \hat{\rho} \hat{M}_m^\dagger . \quad (58)$$

Once again, the conditional and unconditional operators $\hat{\rho}'_m$ [Eq. (55)] and $\hat{\rho}'$ [Eq. (58)] fulfill all the necessary conditions for being legitimate density matrices, see exercise A.

Coming back to the example of the beamsplitter at the end of the last section, the generalization of Eq. (48) to the case of a field initially in a mixed state $\hat{\rho}$ leads to the conditional state

$$\hat{\rho}'_{\text{clac}} = \frac{\hat{a} \hat{\rho} \hat{a}^\dagger}{\text{Tr}(\hat{a} \hat{\rho} \hat{a}^\dagger)} . \quad (59)$$

Equation (58) is very important. While it has been found here as a result of an unread measurement, we will see in Lecture 6 that it is, more generally, the evolution equation of an open system coupled to an environment.

3.4 Use cases of generalized measurements

We finally briefly discuss some use cases of POVMs. As discussed above, many measurements on individual quantum systems cannot be described by PVMs. Instead, POVMs provide us with a suitable framework for describing real measurement processes. For instance, consider again the measurement of an atomic state by position-dependent ionization, as discussed in the introduction of Sec. 3. It can be checked that the operators $\hat{M}_g = |\text{ion}\rangle\langle g|$ and $\hat{M}_e = |\text{ion}\rangle\langle e|$ introduced in Eq. (36) are legitimate Kraus operators. Assume that the atom is in any superposition state $|\psi\rangle = \alpha |g\rangle + \beta |e\rangle$. The atom is supposedly measured in the state $|g\rangle$ with probability $|\alpha|^2$ and in the state $|e\rangle$ with probability $|\beta|^2$. This is indeed what is found using the POVM formula of Eq. (44): $P_g = |\hat{M}_g |\psi\rangle|^2 = |\alpha|^2$ and $P_e = |\hat{M}_e |\psi\rangle|^2 = |\beta|^2$. Moreover, whatever the measurement result, the atom is ionized after the measurement, which is well described by the formula (44):

$$|\psi'\rangle = \frac{\hat{M}_{g/e} |\psi\rangle}{|\hat{M}_{g/e} |\psi\rangle|} = |\text{ion}\rangle . \quad (60)$$

Finally, one can check that the operators $\hat{M}_g$ and $\hat{M}_e$ satisfy the completeness relation (41).

There are many other applications of POVMs. In general, they only provide partial information about the state of the measured system. This is due to the fact that the after-measurement states are generally non-orthogonal. Hence, being in one does not exclude being in the other. An example is provided by the QND measurement in CQED as discussed in Sec. 2.4. when we do not apply the cavity R_2 . In this case, if the atom is measured in $|g\rangle$, the cavity field is projected onto $-i\alpha|0\rangle + \beta|1\rangle$; If it is measured in $|e\rangle$, the cavity field is projected onto $-i\alpha|0\rangle - \beta|1\rangle$. In general, these cavity states are not orthogonal. This case is analyzed in detail in problem B.

Nevertheless, while POVMs only provide partial information on the state of the system, one can build many more Kraus operators – as relevant to POVMs – than projectors – as relevant to PVMs. Indeed, a PVM can only return a number of different results at most equal to the dimension of the Hilbert space of the measured system. On the contrary, a POVM can return a number of different results up to the dimension of the Hilbert space of the ancilla (or the meter).

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A Exercise: After-measurement density matrix

1. (a) Under what conditions is an operator $\hat{\rho}$ a density matrix?
 (b) Show that the operator $\hat{\rho}'$ in Eq. (4) fulfills all the conditions for being a density matrix.
 (c) Show that the conditional and unconditional operators $\hat{\rho}'_{|m}$ and $\hat{\rho}'$ associated to a POVM, Eqs. (55) and (58), fulfill all the necessary conditions for being density matrices.
2. Derive Eq. (6), and show that one recovers the usual Born rule, Eq. (1), in the case of a pure state.
3. Prove Eq. (16) for two systems A and B , i.e. that $P_j = |(\hat{\mathcal{P}}_{j,A} \otimes \hat{I}_B)|\Psi\rangle_{A \otimes B}|^2 = \text{Tr}(\hat{\mathcal{P}}_j \hat{\rho}_A)$ for the bipartite state $|\Psi\rangle_{AB} = \sum_n c_n |\psi_n\rangle_A \otimes |\chi_n\rangle_B$ where the $|\chi_n\rangle$'s form an orthonormal basis of B .

B POVMs in cavity quantum electrodynamics

Consider the cavity quantum electrodynamics experiment discussed in Sec. 2.4. The goal is to realize a QND measurement of the radiation field state in cavity C using a two-level atom A with ground and excited states $|g\rangle$ and $|e\rangle$. Here, we aim at finding the Kraus operators $\hat{M}_g$ and $\hat{M}_e$ associated to the measurement of the atom in either state $|g\rangle$ or $|e\rangle$. With the vocabulary of Sec. 3, the cavity mode plays the role of the system S and the atom that of the meter M . The cavity mode is initially in the state $|\psi\rangle_C = \alpha|0\rangle + \beta|1\rangle$ and the reference state of the atom (meter) is $|e\rangle$.

B.1 Two-cavity apparatus

Consider first the apparatus restricted to only two cavities, i.e. ignore cavity R_2 . We have shown in Sec. 2.4 that the atom-cavity state at the output of cavity C is (see Eq. (32))

$$|\Psi''\rangle_{AC} = -\frac{i\alpha}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |0\rangle + \frac{\beta}{\sqrt{2}}(|g\rangle - |e\rangle) \otimes |1\rangle. \quad (61)$$

1. Find the expressions of $\hat{M}_g(\alpha|0\rangle + \beta|1\rangle)$ and $\hat{M}_e(\alpha|0\rangle + \beta|1\rangle)$. Hint: Write the atom-cavity state (61) in the form $|\Psi\rangle = |\psi_g\rangle_C \otimes |g\rangle_A + |\psi_e\rangle_C \otimes |e\rangle_A$ and identify $\hat{M}_{g/e}(\alpha|0\rangle + \beta|1\rangle)$ as in Eq. (39).
2. Deduce the expressions of the Kraus operators $\hat{M}_g$ and $\hat{M}_e$. Hint: Write the action of these operators onto the basis states $|0\rangle$ and $|1\rangle$.
3. Check the completeness relation, $\hat{M}_g^\dagger \hat{M}_g + \hat{M}_e^\dagger \hat{M}_e = \hat{1}$. Are these Kraus operators Hermitian?
4. Compute the probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the measured atomic state, $P_{0|g}$, $P_{0|e}$, $P_{1|g}$, and $P_{1|e}$.

5. Compute the probabilities P_g and P_e of measuring the atom in $|g\rangle$ or $|e\rangle$. Does the measurement of the atomic state yield information about the photon number?

B.2 Three-cavity apparatus

Consider now the complete three-cavity problem, including cavity R_2 . We found in Sec. 2.4 that the atom-cavity state at the output of cavity R_2 is (see Eq. (34))

$$|\Psi'''\rangle_{AC} = -i\alpha |g\rangle \otimes |0\rangle - \beta |e\rangle \otimes |1\rangle . \quad (62)$$

1. Find the expressions of the operators $\hat{M}_g$ and $\hat{M}_e$, and check the completeness relation. Are these operators Hermitian?
2. Compute the probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the measured atomic state, $P_{0|g}$, $P_{0|e}$, $P_{1|g}$, and $P_{1|e}$.
3. Compute the probabilities P_g and P_e of measuring the atom in $|g\rangle$ or $|e\rangle$. Does the measurement of the atomic state yield information about the photon number?

B.3 Incomplete measurement

We now assume that the third cavity realizes an incomplete $\pi/2$ rotation, corresponding to the transformation

$$\frac{|g\rangle + |e\rangle}{\sqrt{2}} \longrightarrow u |g\rangle + v |e\rangle \quad \text{and} \quad \frac{|g\rangle - |e\rangle}{\sqrt{2}} \longrightarrow v^* |g\rangle - u^* |e\rangle , \quad (63)$$

where $u, v \in \mathbb{C}$ and $|u|^2 + |v|^2 = 1$.

1. Justify that it corresponds to a rotation on the Bloch sphere. It is not required to determine which.
2. Write the atom-cavity state $|\Psi'''\rangle_{AC}$ after R_2 in the form of Eq. (39). Deduce the expressions of the Kraus operators $\hat{M}_g$ and $\hat{M}_e$, and check the completeness relation.
3. Compute the probabilities of measuring the atom in $|g\rangle$ or $|e\rangle$ as a function of $P_0 = |\alpha|^2$ and $\epsilon = |u|^2$. Plot P_g as a function of P_0 for various values of $\epsilon \in [1/2, 1]$.
4. Justify that $\mathcal{E} = |2\epsilon - 1|$ may be called the *efficiency* of the QND measurement. Why do we restrict ourselves to $\epsilon \in [1/2, 1]$? What would be smart to do for $\epsilon \in [0, 1/2]$?