

HW1 - Connection ; Playing with 2-level systems

I Derivations

Pauli Matrices: take $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\begin{smallmatrix} |0\rangle \\ |1\rangle \end{smallmatrix}}$; $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\begin{smallmatrix} |0\rangle \\ |1\rangle \end{smallmatrix}}$

$$1. \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

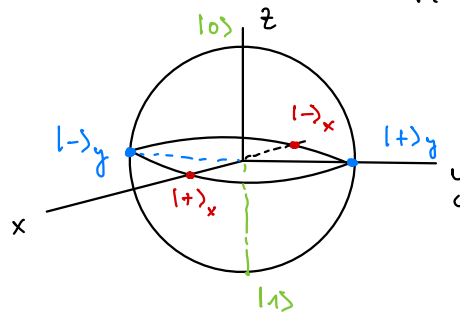
$$2. \text{Tr } \sigma_x = \text{Tr } \sigma_y = \text{Tr } \sigma_z = 0$$

$$3. [\sigma_x, \sigma_y] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_z$$

$$\text{Hence } [\sigma_x, \sigma_y] = 2i\sigma_z ; [\sigma_z, \sigma_x] = 2i\sigma_y ; [\sigma_y, \sigma_z] = 2i\sigma_x$$

$$4. \text{As } \hat{\sigma}_i^2 = 1, \text{ their eigenvalues are such that } \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$5. |\pm\rangle_x = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle) ; |\pm\rangle_y = \frac{1}{\sqrt{2}} (|0\rangle \pm i|1\rangle) ; |\pm\rangle_z = |0\rangle, |1\rangle$$



Spin $\frac{1}{2}$ and Bloch vector

$$\text{Take } \vec{u} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix} \Rightarrow \hat{S} \cdot \vec{u} = \hat{S}_u = S_x \hat{u}_x + S_y \hat{u}_y + S_z \hat{u}_z$$

$$\hat{S}_u = \frac{\hbar}{2} \begin{bmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{bmatrix}$$

$$1. \text{ Calculate } (\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta \Rightarrow \lambda^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\lambda = 1 \Rightarrow \text{Equation of subeqn } (\cos\theta - 1)x + \sin\theta e^{-i\varphi} y = 0$$

$$\text{eigen vector } \parallel \begin{pmatrix} -\sin\theta e^{-i\varphi} \\ \cos\theta - 1 \end{pmatrix} \parallel \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 e^{i\varphi} \end{pmatrix}$$

$$\text{as } \sin\theta = 2\sin\theta/2 \cos\theta/2 \text{ and } 1 - \cos\theta = 2\sin^2\theta/2$$

$$|+\rangle_u = \cos\frac{\theta}{2} |0\rangle + \sin\frac{\theta}{2} e^{i\varphi} |1\rangle$$

And similarly: $|-\rangle_u = \sin \theta/2 |0\rangle - \cos \theta/2 e^{i\varphi} |1\rangle$

$$2. \quad \langle + | \vec{S} | + \rangle_u = \frac{\hbar}{2} \begin{bmatrix} \langle + | \sigma_x | + \rangle_u \\ \langle + | \sigma_y | + \rangle_u \\ \langle + | \sigma_z | + \rangle_u \end{bmatrix}$$

$$\begin{aligned} \text{And: } \langle + | \sigma_x | + \rangle_u &= \begin{pmatrix} \cos \theta/2 & \sin \theta/2 e^{-i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta/2 \\ \sin \theta/2 e^{i\varphi} \end{pmatrix} \\ &= 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2} \right) = \sin \theta \cos \varphi \end{aligned}$$

$$\text{Same calculations yield: } \langle \vec{S} \rangle_u = \frac{\hbar}{2} \begin{bmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{bmatrix} = \frac{\hbar}{2} \vec{u}.$$

Measurement of a qubit: take $\vec{m} = \vec{e}_z$

The state vector associated to $\vec{u}(\theta, \varphi)$ is $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi} |1\rangle$

$$\text{Hence } p_0 = \cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2} \quad \text{and} \quad p_1 = \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$$

$$\text{As } \vec{m} \cdot \vec{u} = \cos \theta \Rightarrow p_{0,1} = \frac{1 \pm \vec{m} \cdot \vec{u}}{2}$$

Schrödinger's Theorem: $\langle A \rangle = \langle \psi(t) | A(t) | \psi(t) \rangle$

$$\frac{d}{dt} \langle A \rangle = \left[\frac{d}{dt} \langle \psi | \right] A | \psi \rangle + \langle \psi | \frac{\partial A}{\partial t} | \psi \rangle + \langle \psi | A \frac{d}{dt} | \psi \rangle$$

$$\text{As } \frac{d}{dt} | \psi \rangle = \frac{1}{i\hbar} H | \psi \rangle \quad \text{and} \quad \frac{d}{dt} \langle \psi | = -\frac{1}{i\hbar} \langle \psi | H$$

$$\begin{aligned} \frac{d}{dt} \langle A \rangle &= \frac{1}{i\hbar} \langle \psi | A H - H A | \psi \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle \\ &= \frac{1}{i\hbar} \langle [A, H] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle \end{aligned}$$

Evolution operator

$$\begin{aligned} U(t) &= \text{Exp} \left[-i \frac{H t}{\hbar} \right] = e^{-i \frac{\Omega t}{2} \sigma_x} \\ &= \sum_{n=0}^{\infty} \frac{\left(-i \frac{\Omega t}{2} \right)^n}{n!} \sigma_x^n = \sum_{p=0}^{\infty} \left(-i \frac{\Omega t}{2} \right)^{2p} \sigma_x^{2p} \frac{1}{(2p)!} + \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left(-i \frac{\Omega t}{2} \right)^{2p+1} \sigma_x^{2p+1} \end{aligned}$$

$$\text{Using } \sigma_x^{2p} = \hat{I} \Rightarrow U(t) = \sum_{p=0}^{\infty} \left(\frac{\Omega t}{2} \right)^{2p} \frac{(-1)^p}{(2p)!} - i \sum_{p=0}^{\infty} \left(\frac{\Omega t}{2} \right)^{2p+1} \frac{(-1)^p}{(2p+1)!} \sigma_x$$

One recognizes the Taylor expansion of $\cos$ and $\sin$.

hence
$$U(t) = \cos \frac{\Omega t}{2} \hat{1} - i \sin \frac{\Omega t}{2} \hat{\sigma}_x$$

Rotation operator $H = \frac{\hbar \Omega}{2} \sigma_x$ has two eigenvalues $\pm \frac{\hbar \Omega}{2}$

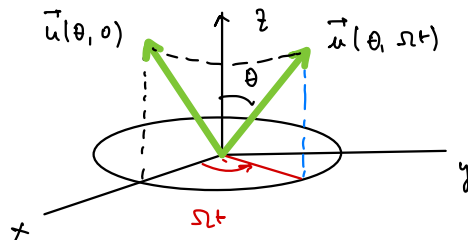
associated to the states $|0\rangle$ and $|1\rangle$

Thus $|\psi(0)\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle \Rightarrow |\psi(t)\rangle = \cos \frac{\theta}{2} e^{-i \frac{\Omega t}{2}} |0\rangle + \sin \frac{\theta}{2} e^{i \frac{\Omega t}{2}} |1\rangle$

or, to an overall phase factor $|\psi(t)\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i \Omega t} |1\rangle$

One recognizes the general expression of a state associated to the

Bloch vector $\vec{u}(\theta, \varphi = \Omega t)$, hence rotated by Ωt with respect to $|\psi(0)\rangle$



II Rabi oscillations

1.
$$i\hbar \begin{pmatrix} \dot{c}_0 \\ \dot{c}_1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \Rightarrow \begin{cases} \dot{c}_0 = i \frac{\Delta}{2} c_0 - i \frac{\Omega}{2} c_1 & (a) \\ \dot{c}_1 = -i \frac{\Omega}{2} c_0 + i \frac{\Delta}{2} c_1 & (b) \end{cases}$$

2. Taking the derivative of (b) $\Rightarrow \ddot{c}_1 + \frac{\Omega^2 + \Delta^2}{4} c_1 = 0$.

3.
$$c_1(t) = A \sin \frac{\sqrt{\Omega^2 + \Delta^2}}{2} t + B \cos \frac{\sqrt{\Omega^2 + \Delta^2}}{2} t$$

As $c_1(0) = 0 \Rightarrow B = 0$

(a) $\Rightarrow \dot{c}_1(0) = -i \frac{\Omega}{2} c_0(0) = -i \frac{\Omega}{2} \times 1 = \frac{\sqrt{\Omega^2 + \Delta^2}}{2} A$

$$c_1(t) = -i \frac{\Omega}{\sqrt{\Omega^2 + \Delta^2}} \sin \left[\sqrt{\Omega^2 + \Delta^2} \frac{t}{2} \right]$$

$$P_1(t) = |c_1(t)|^2 = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2 \left[\sqrt{\Omega^2 + \Delta^2} \frac{t}{2} \right] \text{ which is the Rabi formula}$$

III Zeno effect

1. $\Delta = 0 \Rightarrow P_0(\delta t) = \cos^2\left[\frac{\Omega \delta t}{2}\right] = 1 - \frac{\Omega^2 \delta t^2}{4} + O(\Omega^4 \delta t^4)$
2. $P_0(T) = [P_0(\delta t)]^N \approx \left(1 - \frac{\Omega^2 T^2}{4N^2}\right)^N \approx e^{-\frac{\Omega^2 T^2}{4N}}$
3. $\lim_{N \rightarrow \infty} P_0(T) = 0$
4. If you do not measure every δt , $P_0(T) = \cos^2 \frac{\Omega T}{2}$.

Measuring the system projects it into $|0\rangle$, and prevents the system from evolving: a continuous measurement freezes the state of the system.

IV Entangling gate

$$1. |\psi_{in}\rangle = \frac{1}{\sqrt{2}} \underbrace{(|0\rangle + |1\rangle)}_c \otimes \frac{1}{\sqrt{2}} \underbrace{(|0\rangle + |1\rangle)}_t$$

$$= \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$|\psi_{out}\rangle = U_{\pi}^{(1)} |\psi_{in}\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

2. If not entangled, $\exists \alpha, \beta, \gamma, \delta$ such that:

$$|\psi_{out}\rangle = (\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle)$$

$$\Rightarrow \underbrace{\alpha\gamma = \alpha\delta = \beta\gamma = \frac{1}{2}}_{\alpha^2 \gamma^2 \beta \delta = \frac{1}{8} > 0 \Rightarrow \beta \delta > 0} \quad \text{and} \quad \beta\delta = -\frac{1}{2} \quad \left. \vphantom{\alpha\gamma = \alpha\delta = \beta\gamma = \frac{1}{2}} \right\} \text{incompatible}$$

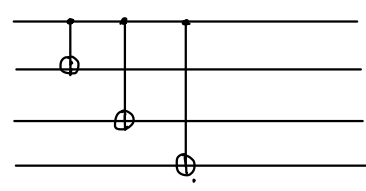
Hence not separable $\Rightarrow$ entangled.

3. $U_{\pi}^{(1)}$ leave the c qubit untouched, hence one just has to

$$\text{look at: } U_H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U_H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$U_{\pi}^{(1)}$ for the target qubit

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4. $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ c  $\frac{1}{\sqrt{2}} (|0 \dots 0\rangle + |1 \dots 1\rangle)$
 $N \times |0\rangle$ c $\left\{ \begin{array}{l} |0\rangle \\ |0\rangle \\ |0\rangle \end{array} \right.$ $N+1$ $N+1$