

Physics of Quantum Information exam (3 hours)

Master 2 QLMN

28 November 2025 (09:00 - 12:00)

ALLOWED documents: 4 pages of personal notes.

NOT ALLOWED: lecture notes and HW, calculators, tablet, smartphone and any electronic device.

Aux francophones (et francographes): Vous pouvez répondre en français.

The exam consists of two independent parts:

1. General, independent questions about the course;
2. A problem on the decoherence of qubits and of entangled states.

Many questions are qualitative and do not require long explanations. Answer them in a concise but precise way.

1 General questions about the course

1. Describe how to implement a qubit in an atomic system, emphasizing in particular (but not only) the following points:
 - How to isolate two states $|0\rangle$ and $|1\rangle$ (especially if they belong to degenerate levels) and what do they correspond to?
 - How to implement a single-qubit gate and what are the control parameters?
2. Explain the principle of a quantum non-demolition (QND) measurement. In particular, what distinguishes a QND measurement from a standard measurement? Give an example of implementation (without detailed description).
3. Write the Lindblad equation. Describe the two parts of the right-hand side and their effect on the state of the system. Show that one of the two parts conserves energy and purity (a demonstration is expected), and argue that the other does not (no demonstration expected).
4. We give the two-qubit state

$$|\psi\rangle = \frac{1}{3}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) + \frac{\sqrt{5}}{6}(|0\rangle - |1\rangle) \otimes (|0\rangle + |1\rangle) . \quad (1)$$

- (a) Is this a Schmidt form? Justify your answer.
- (b) Write the reduced density matrix for the first qubit.
- (c) What is the purity of this state? Comment.

5. Consider the Hamiltonian

$$\hat{H} = \frac{\hbar\Omega}{2}(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger) . \quad (2)$$

- (a) What is the name of this Hamiltonian and what does it describe?

We recall that the eigenstates and eigenenergies of $\hat{H}$ are

$$|n\pm\rangle = \frac{|g, n+1\rangle \pm |e, n\rangle}{\sqrt{2}} \quad \leftrightarrow \quad E_n^\pm = \pm \frac{\hbar\Omega\sqrt{n+1}}{2} . \quad (3)$$

- (b) Assume the system is prepared in the state $|\psi(0)\rangle = |e, 0\rangle$. Compute the state at time t .
- (c) Same question for the initial state $|\psi(0)\rangle = |g, 1\rangle$.
- (d) Explain how to entangle two atoms without direct interaction using a pulse $\pi/2$ for one and a pulse π for the other.

2 Decoherence of a qubit and of entangled states

We analyze in this problem decoherence processes of individual qubits and of collection of qubits prepared in a GHZ state. We recall the expressions of the Pauli operators in the $\{|0\rangle, |1\rangle\}$ basis:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (4)$$

2.1 Spin flips

We assume that the qubit (system $\mathcal{S}$) is coupled to a reservoir $\mathcal{R}$. The Hamiltonian describing the closed system $\mathcal{S} - \mathcal{R}$ leads to a unitary evolution U_{SR} . Initially, the $\mathcal{S} - \mathcal{R}$ state is $|\psi(0)\rangle = |\psi_S\rangle \otimes |\chi_0\rangle$ with $|\chi_0\rangle$ the state of the reservoir.

- 6. We assume that the system can undergo a spin flip around the x or y axis (corresponding to the action of $\sigma_{x,y}$), each with probability $p/2$. Each flip leads to a jump of the state of the reservoir to $|\chi_{x,y}\rangle$. What is the state $U_{SR}|\psi(0)\rangle$ after the flips as a function of p .
- 7. Give the corresponding Kraus operators.
- 8. Use the quantum map to calculate the density matrix ρ' of the qubit after the flips.
- 9. Write the matrix representation of ρ' .

2.2 Dephasing of a qubit

We now suppose that the qubit can only undergo a phase flip (action of σ_z) with a probability p . The state of the reservoir following this is $|\chi_z\rangle$.

10. Starting from the same state as before, $|\psi(0)\rangle = |\psi_S\rangle \otimes |\chi_0\rangle$, what is the state of the system $\mathcal{S} - \mathcal{R}$ after the flip as a function of p ?
11. Give the corresponding Kraus operators.
12. Write the density matrix ρ' corresponding to the quantum map.
13. What is the effect of the dephasing if $p = 1/2$?
14. To understand this, take $|\psi_S\rangle = \alpha|0\rangle + \beta|1\rangle$ and write the state $|\psi_{SR}\rangle$ for $p = 1/2$ in the form $|\psi_{SR}\rangle = \alpha|0\rangle|\phi_0\rangle + \beta|1\rangle|\phi_1\rangle$. Show that $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthogonal, and explain then the loss of coherences.
15. Justify that the jump operator associated to dephasing is $L = \sqrt{\frac{1}{2T_2}} \sigma_z$.
16. Derive the Bloch equations (*i.e.* the differential equations satisfied by ρ_{00} , ρ_{11} , ρ_{01} , and ρ_{10}) from the Lindblad form. Take for the Hamiltonian of the qubit $H = \frac{\hbar\omega_0}{2} \sigma_z$.
17. If the qubit is initially in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, calculate the evolution of the coherence $\rho_{01}(t)$.
18. Give the time scale for the decoherence.
19. Take now N qubits prepared in the GHZ state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N} \right) . \quad (5)$$

and experiencing uncorrelated dephasing, *i.e.* random dephasing from one qubit to another. Considering only the Lindblad part of the master equation, give the scaling timescale of the dephasing of the coherence $\langle 0\dots 0 | \rho | 1\dots 1 \rangle(t)$ with the number N of qubits.

2.3 Collective dephasing of a GHZ state

We consider again a GHZ state of N qubits but we now suppose that the qubits experience a *correlated* dephasing described by the Hamiltonian:

$$H_{\text{noise}} = \frac{\hbar\omega(t)}{2} \sum_{k=1}^N \hat{\sigma}_k^z . \quad (6)$$

20. Propose a situation involving a fluctuating magnetic field where the scenario of correlated noise applies.

21. We want to calculate the fidelity of the state after it has experienced the noise for a duration t , defined as

$$F(t) = \overline{|\langle \psi(0) | \psi(t) \rangle|^2} , \quad (7)$$

where $|\psi(0)\rangle$ is given by Eq. (5) and the bar indicates a statistical average over the fluctuations of the frequency $\omega(t)$. Calculate $|\psi(t)\rangle$ as a function of the quantity $\Phi(t) = \int_0^t \frac{\omega(t')}{2} dt'$.

22. Show that

$$F(t) = \frac{1}{2} \left(1 + \text{Re}[\overline{e^{2iN\Phi(t)}}] \right) \quad (8)$$

23. Assume that the normalized distribution of Φ is Gaussian with a variance $\Delta\Phi$:

$$p(\Phi) = \frac{1}{\sqrt{2\pi}\Delta\Phi} \exp\left(-\frac{\Phi^2}{2\Delta\Phi^2}\right) . \quad (9)$$

Use the integral

$$\int_{-\infty}^{\infty} e^{-u^2\sigma^2} e^{iux} du = \frac{\sqrt{\pi}}{\sigma} e^{-\frac{x^2}{4\sigma^2}} .$$

to show that

$$F(t) = \frac{1}{2} \left(1 + e^{-2\epsilon(N,t)} \right) , \quad (10)$$

and express $\epsilon(N, t)$ as a function of $\Delta\Phi$ and N .

24. The variance $\Delta\Phi$ depends on the time t . Show that for a stationary process, *i.e.* for which $\overline{\omega(t')\omega(t'')}$ depends only on $t' - t''$,

$$\epsilon(N, t) = \frac{N^2}{2} \int_0^t d\tau (t - \tau) \overline{\omega(\tau)\omega(0)} . \quad (11)$$

25. Take a process with a memory time $1/\gamma$ such that $\overline{\omega(\tau)\omega(0)} = \omega_0^2 \exp(-\gamma\tau)$ and show that

$$F(t) = \frac{1}{2} \left(1 + \exp\left[-\frac{1}{\gamma T_2} (e^{-\gamma t} + \gamma t - 1)\right] \right) . \quad (12)$$

Give the expression of T_2 as a function of N , γ and ω_0 .

26. Explain why the Markov approximation corresponds to $\gamma t \gg 1$.
27. Give the expression of $F(t)$ in this case.
28. Plot $F(t)$ and give the scaling of T_2 with N . Compare to the case of uncorrelated noise.

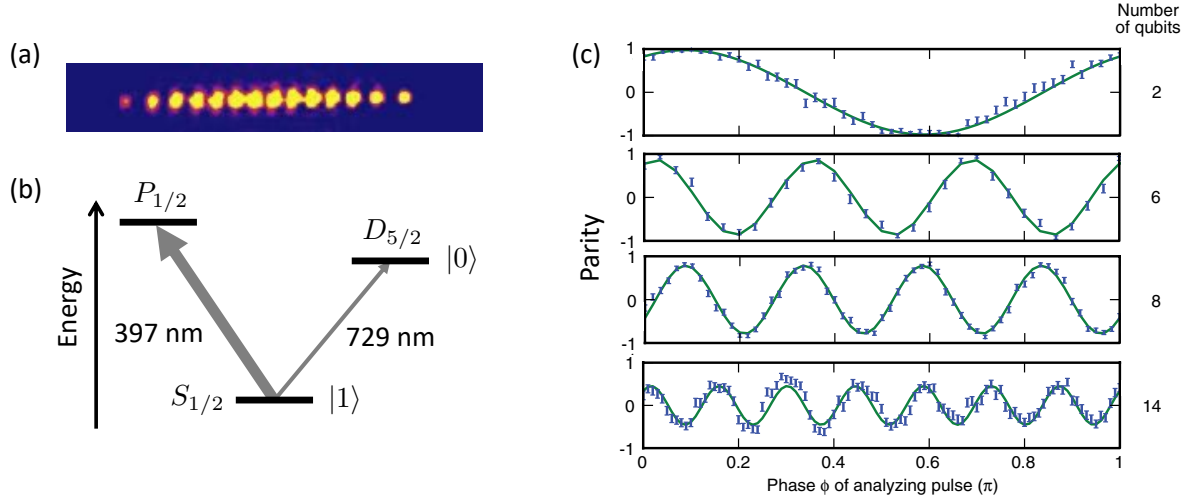


Figure 1: (a) Fluorescence image of a string of 14 ions in a Paul trap. (b) Level structure of the $^{40}\text{Ca}^+$ ion used in the experiment. The readout of the state of the qubit uses the transition at 397 nm, while the qubit is encoded on the quadrupole transition at 729 nm. (c) Parity signal as a function of the phase ϕ of the analyzing pulse, for various numbers of ions N .

2.4 Decoherence of a GHZ state in an ion chain

We describe here an experiment [Phys. Rev. Lett. **106**, 130506 (2011)] that checked the scaling of the decoherence time of a GHZ state predicted for a correlated dephasing. The experiment uses a chain of up to 14 ions trapped in a Paul trap (see Fig. 1a). The qubit is encoded on a quadrupole transition corresponding to a wavelength of 729 nm (see Fig. 1b). The lifetime of the $|0\rangle$ state is 1.16 s. The readout of the qubit relies on a fluorescence measurement using the transition at 397 nm, which has a linewidth of 30 MHz. A sequence of laser pulses applied to the ions prepares an approximation of the GHZ.

29. Describe in a few lines the method used to readout the state of the qubit.
30. To check if the state prepared is close to the GHZ, the authors of the work measured the fidelity of the state at $t = 0$, *i.e.* without letting the state decohere. If the state prepared in the experiment is represented by the density matrix $\hat{\rho}_{\text{exp}}$, show that

$$F(0) = \frac{1}{2}(P_{0N} + P_{1N} + 2\rho_{0N,1N}) , \quad (13)$$

where P_{0N} (resp. P_{1N}) is the probability to find all the qubits in state $|0\rangle$ (resp. $|1\rangle$), and where $\rho_{0N,1N} = \langle 0, \dots, 0 | \hat{\rho}_{\text{exp}} | 1, \dots, 1 \rangle$ is the coherence between $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$ (taken to be real).

31. To measure the coherence, one applies on all the qubits the same rotation

$$\mathcal{R}(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\phi} \\ e^{-i\phi} & 1 \end{pmatrix}_{|0\rangle, |1\rangle} , \quad (14)$$

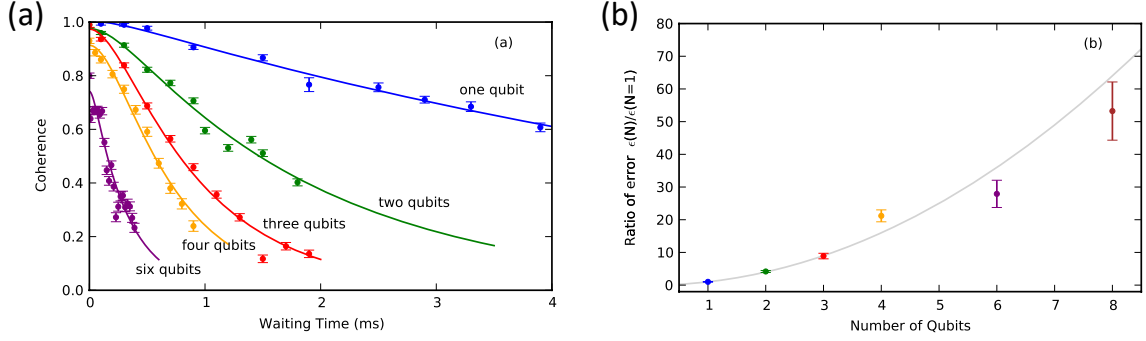


Figure 2: (a) Coherence $\langle 0\dots 0 | \rho | 1\dots 1 \rangle(t)$ of the GHZ state for various numbers of ions. (b) Ratio $\epsilon(N,t)/\epsilon(1,t)$ as a function of the number of ions carrying the qubits.

for a $\pi/2$ pulse before reading the state of the qubits. Considering the levels shown in Fig. 1(a), explain how this can be done.

32. After the rotation, one measures the parity for a given value of the phase ϕ , defined by

$$\Pi(\phi) = \langle \otimes_{j=1}^N \sigma_j^z \rangle . \quad (15)$$

One can show that $\Pi(\phi) = 2\rho_{0N,1N} \cos(N\phi + \delta)$, where δ is a constant phase factor. The results are shown in Fig. 1(c). Explain the signal.

33. Estimate from the data the values of $\rho_{0N,1N}$ for the four ion numbers shown.
34. In a separate measurement, one obtains $P_{0N} + P_{1N} = (0.995, 0.916, 0.847, 0.562)$ for $N = (2, 6, 8, 14)$. Estimate the fidelities and conclude. We remind that a fidelity larger than $1/2$ implies entanglement.
35. To measure the loss of coherence, one applies the rotation $\mathcal{R}(\phi)$ after an evolution time t , and then measures the coherence $\rho_{0N,1N}(t)$. The results are shown in Fig. 2(a). Identify the regime where the Markovian approach is valid, and get an estimate of γ .
36. Use Eq. (12) to explain the behavior at short time.
37. From measurement of the fidelity $F(t)$, one extracts the ratio $\frac{\epsilon(N,t)}{\epsilon(1,t)}$, with results in Fig. 2(b). Do they validate the correlated or uncorrelated dephasing model?
38. In a last set of experiments the authors prepared the 8-qubit state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|00001111\rangle + |11110000\rangle) . \quad (16)$$

Is this state equivalent from the point of view of entanglement to the GHZ state studied above?

39. They repeated the study of the coherence for the new state and found $T_2 \approx 324\text{ms}$ instead of a less than $100\mu\text{s}$ for the GHZ containing 8 ions. Explain why this state can keep its coherence for such a long time.