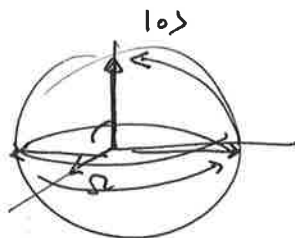


I

1



$$\frac{i}{2} \pi_x \pi_z \frac{\pi}{2} \Rightarrow |0\rangle$$

2. a) a local rotation does not change the amount of entanglement (belong to the same entanglement class)

$$\begin{aligned} b) \quad \rho_{12} &= \text{Tr}_{34} [\underbrace{|\Psi\rangle\langle\Psi|}_\rho] = \langle 00 | \rho | 00 \rangle + \langle 01 | \rho | 01 \rangle + \langle 10 | \rho | 10 \rangle + \langle 11 | \rho | 11 \rangle \\ &= \frac{1}{2} |00\rangle\langle 00| + \frac{1}{2} |11\rangle\langle 11| \end{aligned}$$

$$c) \quad \rho_{12} = \frac{1}{2} |01\rangle\langle 01| + \frac{1}{2} |10\rangle\langle 10|$$

$$d) \quad S = -\frac{1}{2} \ln\left(\frac{1}{2}\right) \times 2 = -\ln 2 = 0.693 \text{ in both cases.}$$

$$3. \quad a) \quad P_0 = \frac{1}{2} \quad P_1 = \frac{1}{2} \text{ in both cases.}$$

b) Measure in a different basis - Example in $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

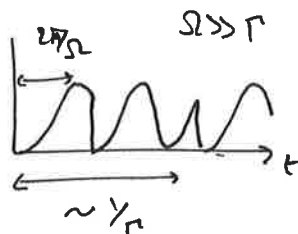
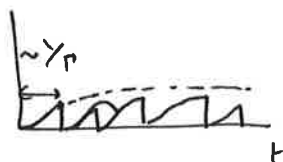
$$\Rightarrow P_+ = 1, P_- = 0 \text{ for the state } \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\begin{aligned} P_+ &= \langle + | \rho | + \rangle = \frac{1}{2} \langle + | 0 \rangle \langle 0 | + \rangle + \frac{1}{2} \langle + | 1 \rangle \langle 1 | + \rangle + \frac{1}{4} \langle + | 0 \rangle \langle 1 | + \rangle \times 2 \\ &= \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} \end{aligned}$$

$$c) \quad \rho^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{1}{16} & \times \\ \times & \frac{1}{4} + \frac{1}{16} \end{pmatrix} \Rightarrow \text{Tr } \rho^2 = \frac{1}{2} + \frac{1}{8} = 0.625$$

4. Open System, entangled with EM field $\Rightarrow$ not a pure state.
Use density operator.

$$5. \quad \Omega \ll \Gamma$$



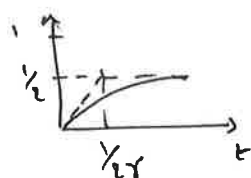
6. a) σ_x is a flip operator transforming $0 \leftrightarrow 1$.

$$b) \quad \dot{\rho} = \frac{1}{i\hbar} \left[\frac{\hbar\omega}{2} \sigma_z \rho \right] + \frac{\gamma}{2} \left[2\sigma_x \rho \sigma_x - \sigma_x^2 \rho - \rho \sigma_x^2 \right]$$

Hence $\dot{\rho} = -i\omega_0 [\sigma_z, \rho] + \gamma [\sigma_x \rho \sigma_x - \rho]$

$$c) \quad \dot{\rho} = -i\frac{\omega_0}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} - \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\ + \gamma \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} - \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\ \dot{\rho} = -i\frac{\omega_0}{2} \left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ -\rho_{10} & -\rho_{11} \end{pmatrix} - \begin{pmatrix} \rho_{00} & -\rho_{01} \\ \rho_{10} & -\rho_{11} \end{pmatrix} \right] + \gamma \begin{bmatrix} \rho_{11} - \rho_{00} & \rho_{10} - \rho_{01} \\ \rho_{01} - \rho_{10} & \rho_{00} - \rho_{11} \end{bmatrix}$$

$$\Rightarrow \quad \dot{\rho}_{11} = \gamma (\rho_{00} - \rho_{11}) \quad ; \quad \dot{\rho}_{01} = -i\omega_0 \rho_{01} + \gamma (\rho_{10} - \rho_{01}) \\ \dot{\rho}_{10} = -i\omega_0 \rho_{10} + \gamma (\rho_{01} - \rho_{10})$$

$$d) \quad \dot{\rho}_{11} = \gamma (1 - 2\rho_{11}) \Rightarrow \rho_{11}(t) = \frac{1}{2} + A e^{-2\gamma t} \\ t=0 \quad \rho_{11}(0) = 0 \Rightarrow \rho_{11}(t) = \frac{1}{2} (1 - e^{-2\gamma t})$$


$$\bullet \quad \dot{\rho}_{01} + \dot{\rho}_{10} = -i\omega_0 (\rho_{01} + \rho_{10}) \Rightarrow \rho_{01} + \rho_{10} = 0 \quad \forall t.$$

e) Markov approx: the Reservoir relaxes rapidly into its ground state after a quantum jump.

7. a) Schmidt decomposition $\Rightarrow |\psi\rangle = \sum_{k=1}^r \sqrt{\lambda_k} |\phi_k\rangle_A |\psi_k\rangle_B$ for a bipartite system A|B.

$$U = e^{-iHt/\hbar} = e^{-iH_A t/\hbar} \otimes e^{-iH_B t/\hbar} = U_A \otimes U_B$$

$$\Rightarrow |\psi\rangle = \sum_{k=1}^r \sqrt{\lambda_k} [U_A |\phi_k\rangle_A] [U_B |\psi_k\rangle_B]$$

The decomposition is unchanged, hence the entanglement remains the same.

b) Not entangled $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$: a measurement $P_A \otimes \mathbb{I}_B$ does not affect B.

• Entangled: the Schmidt decomposition is modified.

Ex: take a Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, measuring $|0\rangle_A$ gives $|0\rangle_B$

II 8. pure state $\Rightarrow \rho^2 = \rho \Rightarrow \text{Tr} \rho^2 = \text{Tr} \rho = 1$

9. Not entangled $\Rightarrow |\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, each part is in a pure state
 $\Rightarrow S_A = S_B = 0$

10. When entangled $\Rightarrow A$ and B are not in a pure state $\Rightarrow S_A < 1$
Hence $S_A = -\ln P_A \geq 0$ Same for B .

11. a) $\langle \hat{P}_{12} \rangle = \text{Tr} [\hat{P}_{12} \rho^{(1)} \otimes \rho^{(2)}] = \sum_m \sum_n \langle m| \langle n| \hat{P}_{12} \sum_{i,j} \rho_{ij}^{(1)} |i\rangle \langle j| \sum_{k,l} \rho_{kl}^{(2)} |k\rangle \langle l| |m\rangle \langle n| \rangle$
 $= \sum_{m,n} \sum_{i,j,k,l} \rho_{ij}^{(1)} \rho_{kl}^{(2)} \langle mn| |j,i\rangle \langle kl| |mn\rangle$
 $= \sum_{m,n} \sum_{i,j,k,l} \rho_{ij}^{(1)} \rho_{kl}^{(2)} \delta_{mi} \delta_{in} \delta_{km} \delta_{ln}$
 $= \sum_{i,j,k,l} \rho_{ij}^{(1)} \rho_{kl}^{(2)} \delta_{jk} \delta_{il} = \sum_{i,k} \rho_{ik}^{(1)} \rho_{ki}^{(2)}$

b) Hence $\langle \hat{P}_{12} \rangle = \sum_i \sum_k \langle i| \rho^{(1)} |k\rangle \langle k| \rho^{(2)} |i\rangle$
 $= \text{Tr} [\rho^{(1)} \rho^{(2)}]$

c) ~~Then~~ $\rho^{(1)} \otimes \rho^{(2)}$ acts in a Hilbert space $2^N = \dim$
while $\rho^{(1)} \rho^{(2)}$ acts in a Hilbert space of $\dim[2^N]$.

12. Take $\rho^{(1)} = \rho^{(2)} \Rightarrow \langle P_{12} \rangle = \text{Tr} [\rho^{(1)2}]$

13. $P_{12}^2 = P_{12} \Rightarrow$ The eigenvalues are such that $P_{12}|\phi\rangle = \lambda|\phi\rangle$
 $\Rightarrow P_{12}^2|\phi\rangle = \lambda^2|\phi\rangle = \lambda|\phi\rangle \Rightarrow \lambda = \pm 1$

14. $\left. \begin{array}{ll} |00\rangle + |11\rangle & \xrightarrow{P_{12}} |00\rangle + |11\rangle \\ |00\rangle - |11\rangle & \rightarrow |00\rangle - |11\rangle \\ |01\rangle + |10\rangle & \rightarrow |10\rangle + |01\rangle \\ |01\rangle - |10\rangle & \rightarrow |10\rangle - |01\rangle \end{array} \right\} \begin{array}{l} +1 \\ -1 \end{array}$

15. $\|\phi^+\rangle\| = \|\phi^-\rangle\| = \|\psi^+\rangle\| = \|\psi^-\rangle\| = 1$

$\langle \phi^+ | \phi^- \rangle = 0 \dots$

Complete as you have 4 vectors for a Hilbert space of $\dim 4$.

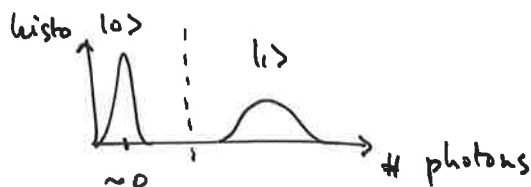
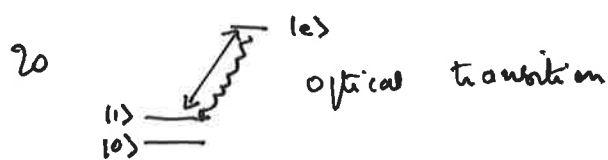
16. Spectral decomposition $\Rightarrow \hat{P}_{12} = |\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-|$ II.2

17. U preserves norm and inner product. $\Rightarrow$ Unitary.

(means that if $\langle\phi_+|\phi_-\rangle = 0 \Rightarrow \langle\phi_+|U^\dagger U|\phi_-\rangle = 0 \dots$)
 $\Rightarrow$ can be constructed by a set of gates.

18. U disentangle the 2 qubits, hence change the Schmidt decomposition.
 The operator $U_A \otimes U_B$ can not change the decomposition, hence the entanglement.

19. $X_{\pi/2}$ can be applied by μ wave rotation, or Raman rotation



21. We don't know how to read directly in the Bell Basis:
 We only know how to measure in the z basis - $|01\rangle |10\rangle |00\rangle |11\rangle$

$$22 \quad |\psi(t)\rangle = (c|0\rangle - is|1\rangle)_A (c|0\rangle - is|1\rangle)_B$$

$$= c^2|00\rangle - ics(|01\rangle + |10\rangle) - s^2|11\rangle$$

$$\text{As } |00\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle) ; \quad |11\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle)$$

$$|\psi(t)\rangle = \frac{c^2}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle) - \frac{s^2}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle) - i\sqrt{2}cs \underbrace{\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)}_{|\phi_+\rangle}$$

$$= \frac{1}{\sqrt{2}}|\psi_-\rangle + \frac{c^2 - s^2}{\sqrt{2}}|\psi_+\rangle - i\frac{\sin 2t}{\sqrt{2}}|\phi_+\rangle$$

$$= \frac{1}{\sqrt{2}}|\psi_-\rangle + \frac{\cos 2t}{\sqrt{2}}|\psi_+\rangle - i\frac{\sin 2t}{\sqrt{2}}|\phi_+\rangle$$

23. from 22 should have $P_{\psi_-} = P_{11} = \frac{1}{2}$

$$P_{\psi_+} = P_{01} = \frac{1}{2}\cos^2 2t$$

$$P_{\phi_+} = P_{10} = \frac{1}{2}\sin^2 2t$$

$$P_{\phi_-} = P_{00} = 0.$$

This is what we observe (up to a phase for ψ_+ / ϕ_+ , but they are indeed π -dephased)

24. Oscillation of P_{01} and P_{10} @ 2Ω .

One reads $T = \frac{2\pi}{2\Omega} = 0,15 \mu s \Rightarrow \Omega/2\pi = \frac{10^6}{0,3} \approx 3 \text{ MHz}$

25. $\langle P_{12} \rangle = \text{Tr}[\rho^1] = P_{\psi_+} + P_{\psi_-} + P_{\phi_+} - P_{\phi_-}$
 $= \underbrace{P_{01} + P_{11} + P_{10}}_{1 - P_{00} \text{ by normalization}} - P_{00} = 1 - 2P_{00}.$

26. $P_{00} = 0,02 \Rightarrow P = 0,96.$

27. If pair i is in 00 the $P_{12} = -1$, $+1$ otherwise

Operator (9) is a product $\Rightarrow \langle P_{12}^{(N)} \rangle = \langle (-1)^{\frac{\# \text{ of } 00}{\uparrow}} \rangle$
 Number of pairs in 00 on a single realization of the expt.

28. The χ_2 chain entanglement entropy $S^2 \uparrow$ Much faster for $|000\rangle$ than $|0101\dots\rangle$: at a given time its value is much larger.

29. After $t=0,5 \mu s$ we observe that S_A depends on the size of the partition (ie its "volume" in 1D) $\Rightarrow$ extensive.

30. Starting from $|0101\dots\rangle$, S_A is nearly independent of the size of the partition n . Theoretically $S_A \propto$ size of boundary which in 1D is 1 $\Rightarrow$ this is what we find expt.

31. Measure independently $S_A^{(1)} / S_B^{(1)}$ following the procedure above and the entropy of the full chain. Construct I_{AB} .

32. If $\rho_{AB} = \rho_A \otimes \rho_B \Rightarrow \rho_{AB}^2 = \rho_A^2 \otimes \rho_B^2$

$$\text{Tr} \rho_{AB}^2 = \text{Tr} \rho_A^2 \times \text{Tr} \rho_B^2 \Rightarrow S_{AB} = -\ln[\text{Tr} \rho_{AB}^2] = -\ln[\text{Tr} \rho_A^2] - \ln[\text{Tr} \rho_B^2]$$

$$= S_A + S_B \Rightarrow I_{AB} = 0.$$

33. • If $I_{AB} \neq 0 \Rightarrow \rho_{AB} \neq \rho_A \otimes \rho_B$ and there exist correlations between A and B (not necessarily entangled though, see q. 35)

• Measuring $S_A^{(1)}$ is not enough as you don't know if AB is in a pure or mixed state. I_{AB} thus compares $S_A + S_B$ to what you have for the full system.

34. For a Bell state $P_A = P_B = \frac{1}{2} \hat{1}$

$\Rightarrow P_A^2 = P_B^2 = \frac{1}{4} \hat{1} \Rightarrow S_A^{(1)} = S_B^{(1)} = \ln 2$

As the Bell state is pure $\Rightarrow S_{AB} = 0$

and $I_{AB}^{(1)} = 2 \ln 2$

35. Being entangled means that ρ can not be decomposed in the form of the convex sum: $\rho = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)}$.

Let us consider a decomposition that is not entangled:

$$P_{18} = \frac{1}{2} \hat{\pi}_{\frac{1}{2}} \otimes \hat{\pi}_{\frac{1}{2}} + \frac{1}{2} (|0 \times 0\rangle \otimes |1 \times 1\rangle) =$$

$$= \frac{1}{8} (|0 \times 0| + |1 \times 1|) (|0 \times 0| + |1 \times 1|) + \frac{1}{2} |0 \times 0|$$

$$= \frac{1}{8} |0 \times 0| + \frac{1}{8} |1 \times 1| + \frac{1}{8} |1 \times 1| + \frac{5}{8} |0 \times 0|$$

$$\text{Tr } \rho_{AB}^2 = \frac{1}{64} \times 3 + \frac{25}{64} = \frac{7}{16} \Rightarrow S_{AB} = -\ln \frac{7}{16} = \ln \frac{16}{7}$$

$$\rho_A = \text{Tr}_B[\rho_{AB}] = \sum_B \langle 0 | \rho_{AB} | 0 \rangle_B + \sum_B \langle 1 | \rho_{AB} | 1 \rangle_B$$

$$= \sum_B \frac{1}{4} (|0\rangle\langle 0| + |1\rangle\langle 1|) \Rightarrow \text{Tr} \rho_A^2 = \frac{9}{16} + \frac{1}{16} = \frac{5}{8}$$

and in the same way, $\text{Tr } \rho_B^2 = 5/8$

Hence $I_{AB}^{(2)} = 2 \ln \frac{8}{5} - \ln \frac{16}{7} \neq 0$

Thus you can be non entangled with $I_{AB} \neq 0$.