

Physics of Quantum Information exam (3 hours)

Master 2 QLMN

29 November 2024 (09:00 - 12:00)

**Allowed documents: 4 pages of personal notes. Calculators allowed.
No lecture notes, no HW, no tablet, no smartphone nor any other
electronic device.**

Aux francophones (et francographes): Vous pouvez répondre en français.

The exam consists of two independent parts:

1. General, independent questions about the course;
2. A problem on the verification of area and volume laws for the entropy of a many-body spin system.

Many questions are qualitative and do not require long explanations. Answer them in a concise but precise way.

1 General questions about the course

1. Use the Bloch sphere to predict the final state of the following sequence: start with the qubit in $|0\rangle$, apply a first $\pi/2$ -pulse around Ox , apply a rotation of angle π around Oz , and apply a last $\pi/2$ -pulse around Ox .
2. Consider the 4-qubit states $|\psi_1\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ and $|\psi_2\rangle = (|0101\rangle + |1010\rangle)/\sqrt{2}$.
 - (a) Explain why the two states belong to the same entanglement class.
 - (b) Calculate the density operator of the first 2 qubits in $|\psi_1\rangle$.
 - (c) Same question for $|\psi_2\rangle$.
 - (d) Calculate the von Neumann entropy of those 2 qubits for both $|\psi_1\rangle$ and $|\psi_2\rangle$. Conclude.
3. Consider the qubit state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and the state described by the density matrix $\hat{\rho} = 0.5 |0\rangle\langle 0| + 0.5 |1\rangle\langle 1| + 0.25 |1\rangle\langle 0| + 0.25 |0\rangle\langle 1|$.
 - (a) Calculate the probability P_0 and P_1 to obtain the result 0 or 1 for both states.

- (b) How would you experimentally differentiate between the two states? Check explicitly that the method yields different results for the two states.
 - (c) Calculate the purity of the state $\hat{\rho}$.
4. Consider an atom driven by a laser and able to spontaneously emit light. Can this atom be described by a pure state, and why? What is the appropriate description and the equation that governs its dynamics?
 5. Take a two-level atom with Γ the linewidth of the excited state. The atom is driven by a laser with Rabi frequency Ω . Use the stochastic wave function approach to plot the evolution of the excited state probability $\pi_e(t)$ as a function of time for a typical quantum trajectory in both cases $\Omega \ll \Gamma$ and $\Omega \gg \Gamma$. Indicate the relevant time scales in the plot.
 6. Consider a qubit with a transition frequency ω_0 . The energy of $|0\rangle$ is $\hbar\omega_0/2$, and that of $|1\rangle$ is $-\hbar\omega_0/2$. The qubit is placed in an environment that can induce spin flips $|0\rangle \leftrightarrow |1\rangle$ with the rate γ .
 - (a) Explain why $\sqrt{\gamma}\hat{\sigma}_x$ is a good Kraus operator.
 - (b) Write the Lindblad equation for the density operator $\hat{\rho}$ using $\hat{\sigma}_z$ and $\hat{\sigma}_x$.
 - (c) Write the differential equations on ρ_{11} and ρ_{01} , and ρ_{10} .
 - (d) Solve them for a qubit initially in the state $|0\rangle$ and plot $\rho_{11}(t)$.
 - (e) Recall the approximation used to derive the Lindblad equation.
 7. Consider a bipartite system $A-B$ and assume that it is submitted to a Hamiltonian of the form $\hat{H} = \hat{H}_A + \hat{H}_B$, where $\hat{H}_A$ acts on the Hilbert space of A and $\hat{H}_B$ on that of B .
 - (a) Write the evolution operator of $A-B$ and show that the entanglement is conserved. You may use the Schmidt decomposition.
 - (b) Assume you perform a PVM on A . Does it affect the state of B ? You may consider both cases where A and B are entangled or not.

2 Checking volume and area laws on a many-body spin system

We discuss here a recent experiment [Bluvstein *et al.*, Nature **604**, 451 (2022)] where the authors measured the second-order Rényi entropy of a chain of spins $1/2$ following an evolution under the Ising Hamiltonian from two different initial states. They were thus able to check that the entropy of the final state after the evolution follows either a so-called area or volume law. In the experiment, the two spin states (qubit) are encoded in two hyperfine states $|0\rangle = |F=1, M=0\rangle$ and $|1\rangle = |F=2, M=2\rangle$ of the ground state of ^{87}Rb atoms (transition frequency $\omega_0/(2\pi) = 6.8\text{ GHz}$). The atoms are each trapped in an optical tweezer and are at rest in their respective traps.

Recall that the second-order Rényi entropy of a state described by a density operator $\hat{\rho}$ is defined as $S^{(2)}(\hat{\rho}) = -\ln(\text{Tr}[\hat{\rho}^2])$, and is thus related to the purity of $\hat{\rho}$, $\mathcal{P} = \text{Tr}[\hat{\rho}^2]$.

2.1 Properties of the entropy

8. Take a pure state $|\psi\rangle$. Give the values of the purity $\mathcal{P}$ and of the entropy $S^{(2)}$.
9. Take a chain of N atoms prepared in a pure state $|\psi_{AB}\rangle$, and split the chain in two parts A (n atoms) and B ($N - n$ atoms). If the two parts are not entangled, how much are the second-order Rényi entropies $S_A^{(2)}, S_B^{(2)}$ of A and B respectively?
10. Suppose now that the two parts are entangled. Explain why $S_A^{(2)} = S_B^{(2)} > 0$.

2.2 Measurement of the second-order Rényi entropy

The method used by the authors relies on the preparation of two copies of the same chain, respectively in states $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$, followed by the measurement of an operator that swaps locally the states of the two corresponding atoms from the two chains (more details below).

The swap operator $\hat{P}_{12}$ is defined as the operator that exchanges the state of two systems 1 and 2:

$$\hat{P}_{12} |\psi\rangle_1 |\phi\rangle_2 = |\phi\rangle_1 |\psi\rangle_2 . \quad (1)$$

11. We first show that for two systems 1 and 2 prepared in the state $\hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}$, $\langle \hat{P}_{12} \rangle = \text{Tr}[\hat{\rho}^{(1)} \hat{\rho}^{(2)}]$.

(a) Write $\hat{\rho}^{(1)} = \sum_{i,j} \rho_{ij}^{(1)} |i\rangle_1 \langle j|_1$ and $\hat{\rho}^{(2)} = \sum_{k,l} \rho_{kl}^{(2)} |k\rangle_2 \langle l|_2$ to show that:

$$\langle \hat{P}_{12} \rangle = \sum_{i,k} \rho_{ik}^{(1)} \rho_{ki}^{(2)} . \quad (2)$$

(b) Conclude that $\langle \hat{P}_{12} \rangle = \text{Tr}[\hat{\rho}^{(1)} \hat{\rho}^{(2)}]$.

(c) What is the difference between the writings $\hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}$ and $\hat{\rho}^{(1)} \hat{\rho}^{(2)}$?

12. How do you need to choose $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$ to access the purity of one of the systems?
13. To measure $\langle \hat{P}_{12} \rangle$ in practice, one first notes that $\hat{P}_{12}^2 = \hat{1}$. Show that the only possible eigenvalues of $\hat{P}_{12}$ are ± 1 .
14. Let us now consider the case of two atoms carrying each a qubit. Calculate the action of $\hat{P}_{12}$ on the four Bell states $|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. Conclude both $+1$ and -1 are indeed eigenvalues of $\hat{P}_{12}$.
15. Show that the set of four Bell states forms a complete orthonormal basis of the two-qubit system, made of maximally entangled states.
16. Conclude that for two qubits

$$\hat{P}_{12} = |\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-| . \quad (3)$$

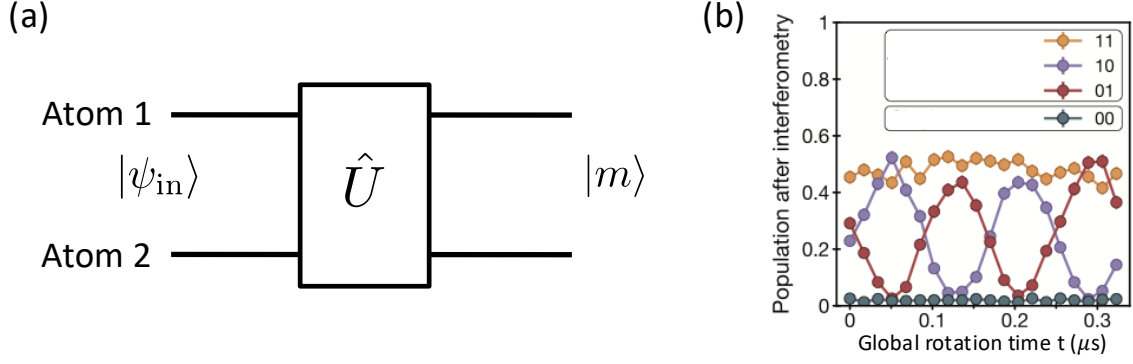


Figure 1: (a) Bell measurement circuit acting on the two-atom state $|\psi_{\text{in}}\rangle$, generating an output state $|m\rangle$. (b) Probability P_{11} , P_{01} , P_{10} , and P_{00} to measure the atoms in state $|11\rangle$, $|01\rangle$, $|10\rangle$ and $|00\rangle$, respectively, after the application of the global rotation on the two qubits represented by the process in (a).

This formula indicates that in order to obtain $\langle \hat{P}_{12} \rangle$ for two qubits, one has to measure the two-atom state in the Bell basis. This can be done applying first a unitary operation $\hat{U}$ (see Fig. 1a), whose action is for instance:

$$|\psi_+\rangle \rightarrow |01\rangle \quad (4)$$

$$|\psi_-\rangle \rightarrow |11\rangle \quad (5)$$

$$|\phi_+\rangle \rightarrow |10\rangle \quad (6)$$

$$|\phi_-\rangle \rightarrow |00\rangle, \quad (7)$$

and then measuring in the product basis. The operation means that, for example, when the input state $|\psi_{\text{in}}\rangle = |\psi_+\rangle$, the first atom is in state $|0\rangle$, the second one in state $|1\rangle$ after the application of $\hat{U}$.

17. Show that $\hat{U}$ is unitary and conclude that it can be constructed (we do not ask an explicit circuit).
18. Justify that $\hat{U}$ cannot take the form $\hat{U}_A \otimes \hat{U}_B$ and requires non-separable two-qubit operations. You may recall what you have done at question 7.
19. The circuit building $\hat{U}$ also contains single-qubit rotations such as $X_{\pi/2}$. Explain how you can apply such a rotation in practice.
20. Explain how to measure if an atom is in state $|0\rangle$ or $|1\rangle$ using a laser and an extra atomic level.
21. Why not performing a measurement directly in the Bell basis ?

To check that the method works, the authors first performed an experiment with two independent atoms, each encoding a qubit. They apply on each of the atom initially prepared in $|0\rangle$ the same global rotation: $|0\rangle \rightarrow \cos\left(\frac{\Omega t}{2}\right) |0\rangle - i \sin\left(\frac{\Omega t}{2}\right) |1\rangle$.

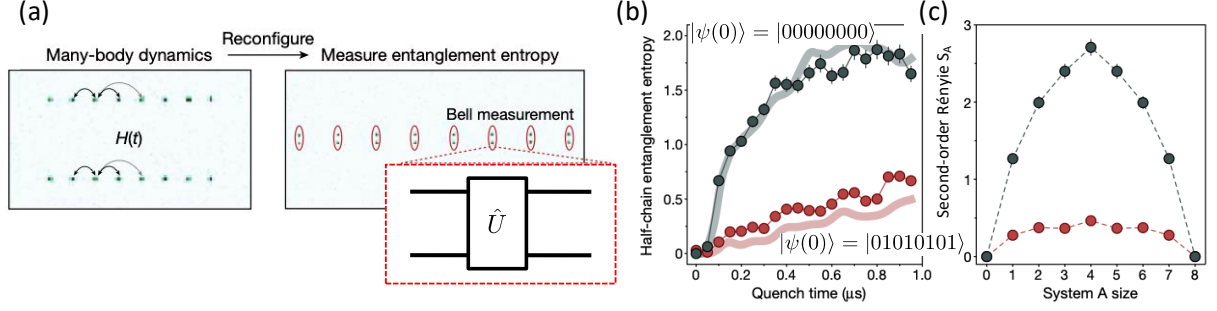


Figure 2: (a) Principle of the experiment to measure the second-order Rényi entropy of a chain: (i) prepare two independent chains both in $|\psi(0)\rangle$; (ii) let them evolve under the Hamiltonian (10); (iii) bring the two chains close to each other by moving the atoms; and (iv) perform a Bell measurement on each pair i . (b) $S_A^{(2)}(t)$ of the first half A of the chain measured after an evolution time t for the two initial states. (c) $S_A^{(2)}$ measured after a $t = 0.5 \mu\text{s}$, where A contains n atoms and B , $N - n$.

22. Show that after applying the rotation for a time t , the two qubit state is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} |\psi_{-}\rangle + \frac{\cos(\Omega t)}{\sqrt{2}} |\psi_{+}\rangle - i \frac{\sin(\Omega t)}{\sqrt{2}} |\phi_{+}\rangle \quad (8)$$

23. After this global rotation, the authors measured the 2-qubit states in the Bell basis using the circuit of Fig. 1(a). The measured probabilities are shown in Fig. 1(b). Explain and comment the results in light of the previous questions.

24. Extract the value of the Rabi frequency $\Omega/(2\pi)$.

25. Show that the purity of the state of one of the two atoms is $1 - 2P_{00}$, where P_{00} is the probability to find $|00\rangle$ after application of the circuit.

26. Experimentally, $P_{00} \approx 0.02$. Calculate the purity.

We now generalize to a chain of N atoms, both prepared in the same state $\hat{\rho}$. The two chains are then brought close to each other by moving the atoms, as shown in Fig. 2(a), and the Bell measurement circuit is applied on each pair i , thus implementing the global swap operator:

$$P_{12}^N = \otimes_{i=1}^N P_{12}^{(i)} . \quad (9)$$

27. Explain why $\langle \hat{P}_{12}^N \rangle = \langle (-1)^{N_{00}} \rangle = \text{Tr}[\hat{\rho}^2]$, where N_{00} is the number of pairs 00 obtained at the end of the Bell measurement.

2.3 Checking area and volume laws

The authors of the experiment then prepared two chains of 8 atoms in the same initial state for two cases: either $|\psi(0)\rangle = |00000000\rangle$ or $|\psi(0)\rangle = |01010101\rangle$. The two chains are initially far from each other so that they are completely independent. Using an

experimental procedure that we do not explain here, the atoms in the two chains evolve independently under the same transverse field Ising Hamiltonian:

$$\hat{H} = \sum_{i=1}^{N-1} V_{i,i+1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \frac{\hbar\Omega}{2} \sum_{i=1}^N \hat{\sigma}_i^x - \hbar\Delta \sum_{i=1}^N \hat{\sigma}_i^z, \quad (10)$$

where V_{ij} is the interaction energy between two atoms i and j . The state after the evolution under $\hat{H}$ is $|\psi(t)\rangle = \exp[-i\hat{H}t/\hbar] |\psi(0)\rangle$. Following this evolution, the authors applied the procedure described above to measure the second-order Rényi entropy of half the chain: they considered the purity of the first part of the chain, containing $n = 4$ atoms, and plotted this quantity for various evolution times t and for the two different initial states. The results are shown in Fig. 2(b). We observe a very different time evolution for the two initial states.

28. Explain briefly why one concludes that the entanglement grows much faster for the initial state $|\psi(0)\rangle = |00000000\rangle$ than for $|01010101\rangle$.
29. To check the scaling of the entropy with the size of the subsystem A containing n atoms, the authors measure the Rényi entropy after an evolution time $t = 0.5 \mu\text{s}$, for different bipartitions of the chain (of size n and $N - n$). Figure 2(c) shows the result for the two different initial states. Explain why one says that the state $|\psi(0)\rangle = |00000000\rangle$ leads to a volume law.
30. For an area law, the entropy is proportional to the area of the surface separating the bipartition: for a one dimensional system, this should be 1. Explain why the data confirm that the state $|01010101\rangle$ leads to an area law.

2.4 Mutual information (bonus questions)

In fact the quantity plotted in Fig. 2(c) is not $S_A^{(2)}$, but the so-called mutual information

$$I_{AB}^{(2)} = S_A^{(2)} + S_B^{(2)} - S_{AB}^{(2)}, \quad (11)$$

where $S_{AB}^{(2)}$ is the entropy of the full chain.

31. Explain how it is measured experimentally.
32. What is the value of $I_{AB}^{(2)}$ in the case where the density matrix factorizes, $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$?
33. Conclude that $I_{AB}^{(2)} \neq 0$ signals correlations between A and B , and explain why the authors considered $I_{AB}^{(2)}$ instead of simply $S_A^{(2)}$.
34. For instance, what is the value of $I_{AB}^{(2)}$ for a Bell state?
35. Do you think that $I_{AB}^{(2)} \neq 0$ signals entanglement? You may consider the case where

$$\hat{\rho}_{AB} = \sum_{k=1,2} p_k \hat{\rho}_A^{(k)} \otimes \hat{\rho}_B^{(k)}, \quad (12)$$

with $p_1 = p_2 = 1/2$, $\hat{\rho}_A^{(1)} = \hat{\rho}_B^{(1)} = \hat{I}/2$, $\hat{\rho}_A^{(2)} = |0\rangle\langle 0|$, and $\hat{\rho}_B^{(2)} = |1\rangle\langle 1|$.