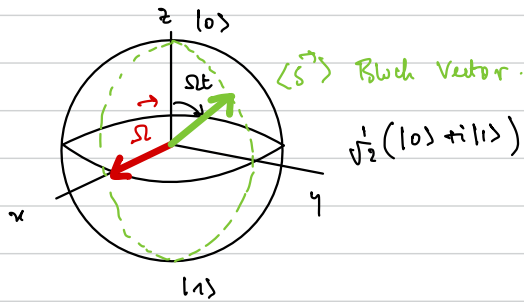


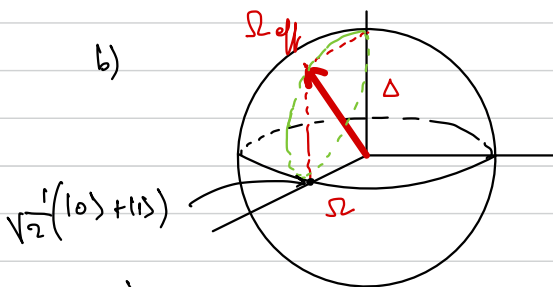
Exam Physics of Quantum Information - Solution

I 1.



The Bloch vector precesses around the $\hat{x}$ axis with an angular frequency Ω

2. a) from lecture 1 $\vec{\Omega} = \begin{pmatrix} \Omega \\ 0 \\ -\Delta \end{pmatrix}$ (The sign $-\Delta$ or $+\Delta$ is non essential)



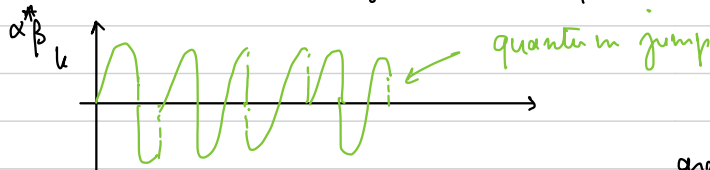
The Bloch vector precesses around the axis $\hat{x} + \hat{z} \parallel \vec{\Omega}$

The Rabi frequency is $\Omega_{eff} = \sqrt{\Omega^2 + \Delta^2}$

- c) The precession brings the Bloch vector from $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ in a time T such that
- $$\Omega_{eff} T = \pi \Rightarrow T = \frac{\pi}{\sqrt{\Omega^2 + \Delta^2}}$$

3. In the stochastic wave function approach $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle$, and $\Rightarrow P_{eg} = \alpha^* \beta$ and the average dipole is $\langle d \rangle \propto P_{eg}$

For a given quantum trajectory (k) for which $\Omega \gg \Gamma$.



The average time between jumps is $1/\Gamma \Rightarrow$ there are many Rabi cycles between.

2 jumps $\Rightarrow$ The average over many trajectories $\rightarrow 0$

4. a) Yes as $(\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{\sqrt{2}})^2 = 1$

- b) Yes. Otherwise $|\psi_{SE}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$
and $\alpha\gamma = \frac{1}{2}$, $\alpha\delta = \frac{1}{2}$, $\beta\gamma = \frac{1}{\sqrt{2}}$, $\beta\delta = 0$.

Incompatible to fulfill at the same time.

- c) $P_S = \sum_E \langle 0 | \psi_{SE} \rangle \langle \psi_{SE} | 0 \rangle + \sum_E \langle 1 | \psi_{SE} \rangle \langle \psi_{SE} | 1 \rangle$

$$\sum_E \langle 0 | \psi_{SE} \rangle = \frac{1}{2} |0\rangle_S + \frac{1}{\sqrt{2}} |1\rangle_S \quad \text{and} \quad \sum_E \langle 1 | \psi_{SE} \rangle = \frac{1}{2} |0\rangle_S$$

$$\Rightarrow P_S = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| + \frac{1}{2\sqrt{2}} (|0\rangle \langle 1| + |1\rangle \langle 0|) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

5. a) basis $|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

$$b) \cdot \rho_A^W = \sum_{B,C} \langle 00|W\rangle \langle W|00\rangle + \langle 01|W\rangle \langle W|01\rangle + \langle 10|W\rangle \langle W|10\rangle + \langle 11|W\rangle \langle W|11\rangle$$

$$\text{As } \langle 00|W\rangle = \frac{1}{\sqrt{3}} \langle 1|_A \quad \langle 01|W\rangle = \frac{1}{\sqrt{3}} \langle 0|_A \quad \langle 10|W\rangle = \frac{1}{\sqrt{3}} \langle 0|_A \quad \langle 11|W\rangle = 0$$

$$\rho_A^W = \frac{1}{3} |1\rangle\langle 1|_A + \frac{2}{3} |0\rangle\langle 0|_A$$

$$\cdot \rho_A^{GHZ} = \langle 00|GHZ\rangle \langle GHZ|00\rangle + \langle 01|GHZ\rangle \langle GHZ|01\rangle + \langle 10|GHZ\rangle \langle GHZ|10\rangle + \langle 11|GHZ\rangle \langle GHZ|11\rangle$$

$$\text{As } \langle 00|GHZ\rangle = \frac{1}{\sqrt{2}} \langle 0|_A \quad \langle 01|GHZ\rangle = 0 = \langle 10|GHZ\rangle \quad \langle 11|GHZ\rangle = \frac{1}{\sqrt{2}} \langle 1|_A$$

$$\Rightarrow \rho_A^{GHZ} = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1|$$

c) $S = - \sum_{\lambda} \lambda \log \lambda$ where λ are the eigenvalues of ρ . Here both matrices are already diagonal

$$\text{Hence: } S_W = - \frac{1}{3} \ln \frac{1}{3} - \frac{2}{3} \ln \frac{2}{3} = 0,636$$

$$S_{GHZ} = - \frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \ln \frac{1}{2} = 0,693.$$

d) The most entangled state has the largest entropy of a partition
hence GHZ is more entangled than W .

6. a) Empty cavity + atom in the ground state $\rightarrow$ no modification
 $|g,0\rangle \rightarrow |g,0\rangle$

$$b) \text{ Recall } |e\rangle|0\rangle \rightarrow [M_0|e\rangle]|0\rangle + [M_1|e\rangle]|1\rangle$$

$$\text{by identification } M_0|e\rangle = \frac{1}{\sqrt{2}}|e\rangle \text{ and } M_1|e\rangle = -\frac{i}{\sqrt{2}}|g\rangle$$

$$|g,0\rangle \rightarrow [M_0|g\rangle]|0\rangle + [M_1|g\rangle]|1\rangle$$

$$\text{hence: } M_0|g\rangle = |g\rangle \text{ and } M_1|g\rangle = 0$$

$$\text{Thus: } M_0 = \frac{1}{\sqrt{2}} |e\rangle\langle e| + |g\rangle\langle g|$$

$$c) M_1 = -\frac{i}{\sqrt{2}} |g\rangle\langle e|$$

$$d) M_0^\dagger M_0 + M_1^\dagger M_1 = \left(\frac{1}{\sqrt{2}} |e\rangle\langle e| + |g\rangle\langle g| \right) \left(\frac{1}{\sqrt{2}} |e\rangle\langle e| + |g\rangle\langle g| \right) + \frac{1}{2} |e\rangle\langle g| |g\rangle\langle e|$$

$$= |e\rangle\langle e| + |g\rangle\langle g| = \hat{\mathbb{I}}$$

7. a)
- Prepare the atom in $|0\rangle$
 - Apply laser for a time t fixed
 - Read out : you find 0 or 1 (Example make the atom fluoresces if in state $|0\rangle$, not if in $|1\rangle$)
 - Repeat the procedure N times and count N_0, N_1
 The $p_0 = \frac{N_0}{N}$, $p_1 = \frac{N_1}{N}$
 - Change t and repeat the above sequence.

b) Rabi frequency: $P_1(t) = \sin^2 \frac{\Omega t}{2}$

c) Period $T = 0,4 \mu s \Rightarrow \frac{\Omega}{2\pi} = \frac{1}{T} = 2,5 \text{ MHz}$

d) $\tilde{P}_1(t) = (1-\epsilon) \times P_1(t)$

↑
Proba that $|1\rangle$ is actually $|1\rangle$ and not $|0\rangle$

e) $1-\epsilon$ is thus the largest value of the measured probability $\tilde{P}_1$

One reads $1-\epsilon \approx 0,9 \Rightarrow \epsilon \approx 0,1$

II 8. $H = -\frac{\hbar\omega_0}{2} \sigma_z$ describes the atom

$\sum_k \hbar\omega_k a_k^\dagger a_k$ hamiltonian of the field mode. (quantized)

$\sum_k \hbar g_k (a_k^\dagger \sigma^- + a_k \sigma^+)$ is the atom / quantized field interaction $-\vec{d} \cdot \vec{E}$

The fact that $a_k^\dagger \sigma^+$ and $a_k \sigma^-$ are not present come from the resonant approximation.

9. $|e\rangle : +\frac{\hbar\omega_0}{2}$

$|g|k\rangle : -\frac{\hbar\omega_0}{2} + \hbar\omega_k$

10. Schrödinger: $i\hbar \left[(\dot{c}_e - i\frac{\omega_0}{2} c_e) e^{-i\omega_0 t/2} |e\rangle + \sum_k \dot{c}_k e^{-i(\omega_k - \frac{\omega_0}{2})t} |g|k\rangle \right] = H |\psi\rangle$

Projection $\langle e| \Rightarrow \dot{c}_e = -i \sum_k g_k e^{-i(\omega_k - \omega_0)t} c_k(t)$

$\langle g|k| \Rightarrow \dot{c}_k = -i g_k e^{i(\omega_k - \omega_0)t} c_e(t)$

11. $c_k(t) - 0 = -i g_k \int_0^t c_e(t') e^{i(\omega_k - \omega_0)t'} dt'$

$\Rightarrow \dot{c}_e(t) = - \sum_k g_k^2 \int_0^t c_e(t') e^{i(\omega_k - \omega_0)(t-t')} dt'$

12. Replacing $\sum_k$ by $\int d\omega_k \rho(\omega_k)$ leads to:

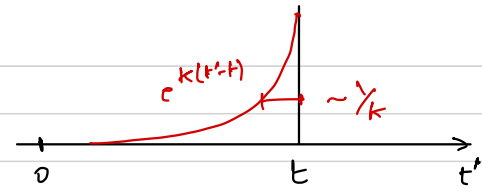
$\dot{c}_e(t) = - \int_0^t c_e(t') \int \rho(\omega_k) g_k^2 e^{i(\omega_k - \omega_0)(t-t')} d\omega_k dt'$

But $g_k^2 \propto \omega_k$ and $\rho(\omega_k)$ is peaked around ω_0 with a width $k \ll \omega_0 \Rightarrow$ can assume that $g_k \sim g_0 = d \sqrt{\frac{\omega_0}{\hbar \epsilon_0 V}}$

13. then $\dot{c}_e(t) = - g_0^2 \int_0^t c_e(t') \frac{1}{\pi k} \underbrace{\int_0^\infty \frac{d\omega_k}{1 + \frac{(\omega_k - \omega_0)^2}{k^2}} e^{i(\omega_k - \omega_0)(t-t')} d\omega_k}_{\pi k e^{k(t-t')}} dt'$

$= - \underbrace{\left(\frac{2g_0^2}{k} \right)}_{\Gamma_c} \frac{1}{2} k \int_0^t c_e(t') e^{k(t-t')} dt'$

14. The function $e^{k(t-t')}$ for $t' < t$ looks like:



The width is $\sim 1/k$. If c_e varies slowly in a time $1/k$, then it can be considered constant under the integral

$$\Rightarrow \dot{c}_e \sim - \frac{\Gamma_c k}{2} c_e(t) \int_0^t e^{k(t-t')} dt'$$

$$\frac{1}{k} \left[e^{k(t-t')} \right]_0^t = \frac{1}{k} (1 - e^{-kt}) \sim 0 \text{ for } t \gg \frac{1}{k}$$

$$\dot{c}_e(t) \sim - \frac{\Gamma_c}{2} c_e(t)$$

As $\Gamma_c \ll k$, the timescale for the variation of c_e is $\sim 1/\Gamma_c$, much longer than $1/k$ and the procedure above is valid...

15. $c_e(t) = c_e(0) e^{-\Gamma_c/2 t}$ and $c_e(0) = 1$

16. From Eq (7): $\ddot{c}_e(t) = - \frac{\Gamma_c k}{2} c_e(t) e^{k(t-t)} - k \left(- \frac{\Gamma_c k}{2} \int_0^t c_e(t') e^{k(t-t')} dt' \right)$

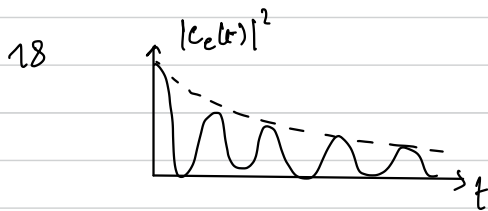
[Recall that $\frac{d}{dx} \int_0^x f(x,t) dt = f(x,x) + \int_0^x \frac{\partial f}{\partial x}(x,t) dt]$ $\dot{c}_e(t)$

Hence: $\ddot{c}_e + k \dot{c}_e + \frac{\Gamma_c k}{2} c_e = 0$

17. If $k \ll \Gamma_c$ $k' = k \sqrt{-\frac{2\Gamma_c}{k}} = i g_0$

and $c_e(t) = e^{-k t/2} \cos \frac{g_0 t}{2}$ or $|c_e(t)|^2 = e^{-k t} \cos^2 \frac{g_0 t}{2}$

Damped, single photon oscillation at Rabi frequency g_0 .



19. • for $k \ll \Gamma_c$, the photon stays in the cavity for a long time ($1/k$) it is periodically absorbed by the atom.

• for $k \gg \Gamma_c$ on the contrary, it leaves the cavity before being absorbed: one expects an exponential decay $e^{-\Gamma_c t}$

(Broad continuum...)

$$20. \quad \mathcal{F} = \frac{2g_0^2}{\kappa} \frac{1}{\Gamma} = 2 \frac{\deg^2 \omega_0}{\hbar \epsilon_0 V \kappa} \frac{1}{\frac{\deg^2 \omega_0^3}{3\pi \epsilon_0 c^3}} = \frac{6\pi}{8\pi^3} Q \frac{\lambda^3}{V}$$

$$\mathcal{F} = \frac{3}{4\pi^2} Q \frac{\lambda^3}{V}$$

$$21. \quad \mathcal{F} = \frac{3}{4\pi^2} 2000 \left(\frac{0,9}{3,5}\right)^3 10^{-18} \frac{1}{10^{-19}} = 25.$$

$$22. \quad \beta = \frac{25}{26} = 0,96 \quad \text{It means that } 96\% \text{ of the photons are emitted in the cavity mode, i.e. exist the cavity in a well defined direction rather than in } 4\pi \text{ sr.}$$

Can thus be efficiently collected, eg. in optical fibers.