

Physics of Quantum Information exam (2 hours 45)

Master 2 QLMN

02 December 2022 (09:00 - 11:45)

Allowed documents : lecture notes and problem sets. Calculators allowed.

Aux francophones (et francographes) : Vous pouvez répondre en français.

The exam consists of two independent parts :

1. General independent questions about the course ;
2. A problem on the coupling of an atom to a continuum with a variable width.

1 General questions about the course

1. Represent on the Bloch sphere a Rabi oscillation at resonance, around the $\hat{x}$ axis, for a qubit initially in $|0\rangle$ (north pole of the Bloch sphere).
2. Consider a qubit driven by a wave with Rabi frequency Ω and a detuning Δ with respect to the qubit transition.
 - (a) What are the coordinates of the effective field $\mathbf{\Omega}$ in the Bloch sphere representation ?
 - (b) Consider the case $|\Delta| = \Omega$: show that if the qubit is initially in $|0\rangle$, it is possible to prepare the state $(|0\rangle + |1\rangle)/\sqrt{2}$.
 - (c) What should be the duration T of application of the wave for this preparation ?
3. Consider a two-level atom (states $|g\rangle$ and $|e\rangle$) driven by a resonant laser with Rabi frequency Ω . The linewidth of the excited state $|e\rangle$ is Γ . We consider the case $\Omega \gg \Gamma$. Use the stochastic wave function picture to explain why the steady-state coherence ρ_{eg} is nearly zero, and therefore the average dipole is zero.
4. Consider two coupled systems S and E in the state

$$|\Psi\rangle_{SE} = \frac{1}{2} |0\rangle_S \otimes (|0\rangle + |1\rangle)_E + \frac{1}{\sqrt{2}} |1\rangle_S \otimes |0\rangle_E.$$

- (a) Is this vector state normalized ?
- (b) Is it an entangled state ?
- (c) Calculate the density operator of S .

5. Consider 3 qubits A, B and C. They can be prepared either in $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ or in $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. We want to tell which one is more entangled, using the entanglement entropy criterion. To do so, we split the system in a bi-partition A and (B+C)
 - (a) Give a basis describing the (B+C) system.
 - (b) Calculate the density operator of the qubit A for both the W and GHZ states.
 - (c) Calculate the von Neumann entanglement entropies S_W and S_{GHZ} of A for both the W and GHZ states.
 - (d) Conclude on which of the two states is the most entangled.
6. Consider an experimental situation analogous to the one of Lecture 5 : A two-level atom crosses a cavity, which is initially empty. We will consider here the cavity as the measuring device and the atom as the object to be measured. We set the atom-cavity interaction time so that if the atom is initially in state $|e\rangle$ and the cavity initially empty $|n=0\rangle$, their state becomes $(|e, 0\rangle - i|g, 1\rangle)/\sqrt{2}$ after the atom has crossed the cavity.
 - (a) If the atom-cavity state is initially $|g, 0\rangle$, what is the state after the atom has crossed the cavity?
 - (b) Calculate the Kraus operator $\hat{M}_0$ acting on the atom corresponding to the cavity being empty after the atom has crossed the cavity.
 - (c) Same question for $\hat{M}_1$ corresponding to the cavity containing 1 photon after the atom has crossed the cavity.
 - (d) Check that the condition $\hat{M}_0^\dagger \hat{M}_0 + \hat{M}_1^\dagger \hat{M}_1 = \mathbb{1}$ is fulfilled.
7. Consider an atomic qubit carried by a *single* ^{87}Rb atom, laser-cooled and trapped in an optical tweezer. The qubit states are $|0\rangle$ and $|1\rangle$. The atom is driven by a laser resonant with the qubit frequency and with a Rabi frequency Ω . The experimentalists measure the Rabi oscillation for an atom initially in $|0\rangle$ and obtain the curve shown in the figure ??.

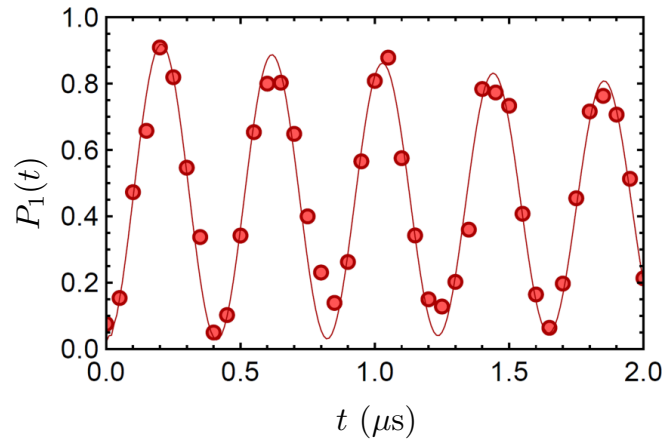


FIGURE 1 – Rabi oscillation measured on a single atom trapped in an optical tweezer.

- (a) Explain how the curve was measured in practice.
- (b) Give without demonstration the probability $P_1(t)$ as a function of Ω and t .
- (c) Give the value of the experimental Rabi frequency $\Omega/(2\pi)$ in MHz.
- (d) In practice the detection of state $|1\rangle$ is imperfect : A qubit in $|1\rangle$ is actually found in state $|0\rangle$ with probability ϵ . Calculate the measured probability $\tilde{P}_1$ as a function of the true probability P_1 and ϵ .
- (e) Estimate ϵ from the figure.

2 Continuum of variable width. Application to the Purcell effect

We consider a 2-level atom (state $|g\rangle$ and $|e\rangle$, transition frequency ω_0) placed in a cavity formed by two mirrors facing each other. One of the two mirrors is perfectly reflecting, while the other mirror has a transmission coefficient $T \ll 1$ to allow for the light to come out of the cavity : The cavity thus has a loss rate κ . The atom initially in $|e\rangle$ can decay into $|g\rangle$ by populating the cavity modes k , with a rate Γ_c , which we will calculate. The goal of the problem is to find the time evolution of the atomic state depending on the ratio Γ_c/κ .

The Hamiltonian describing the problem in the quasi-resonant approximation is

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_k \hbar g_k \left(\hat{\sigma}_+ \hat{a}_k + \hat{\sigma}_- \hat{a}_k^\dagger \right) , \quad (1)$$

with $\hbar g_k = d_{eg} \sqrt{\hbar\omega_k/(\epsilon_0 V)}$ and V the mode volume.

- 8. Explain briefly what the various terms in the Hamiltonian describe.
- 9. The cavity being initially empty, we consider as a basis to describe the problem the states $|e, 0\rangle$ and $\{|g, 1_k\rangle\}_k$ where 1_k corresponds to one photon in the mode k . Give the energy of the two states for $g_k = 0$.
- 10. We write the atom-cavity state as

$$|\psi(t)\rangle = c_e(t) e^{-i\frac{\omega_0 t}{2}} |e, 0\rangle + \sum_k c_k(t) e^{-i(\omega_k - \frac{\omega_0}{2})t} |g, 1_k\rangle . \quad (2)$$

Give the differential equations fulfilled by $c_e(t)$ and $c_k(t)$.

- 11. Integrating formally the equations for $\dot{c}_k$ [with $c_k(t=0) = 0$] and inserting them into the equation for $\dot{c}_e$, show that

$$\dot{c}_e = - \int_0^t \sum_k g_k^2 e^{i(\omega_k - \omega_0)(t'-t)} c_e(t') dt' . \quad (3)$$

- 12. We now assume that the frequencies ω_k of the cavity modes are very close to each other so as to form a quasi-continuum of width κ , and that they are distributed according to the distribution

$$\rho(\omega_k) = \frac{1}{\pi\kappa} \frac{1}{1 + (\omega_k - \omega_0)^2/\kappa^2} . \quad (4)$$

This holds for a cavity tuned near the atomic resonance. We can then replace $\sum_k$ by $\int_0^\infty d\omega_k \rho(\omega_k)$ in Eq. (??). For $\omega_0 \gg \kappa$, explain why g_k can be approximated by

$$g_0 = d_{eg} \sqrt{\frac{\omega_0}{\hbar \epsilon_0 V}} . \quad (5)$$

13. Using the formula

$$\int_{-\infty}^{\infty} dx \frac{e^{ixt}}{1+x^2/a^2} = \pi a e^{-at} \quad \text{for } t > 0 , \quad (6)$$

show that

$$\dot{c}_e(t) \approx -\frac{\Gamma_c \kappa}{2} \int_0^t c_e(t') e^{\kappa(t'-t)} dt' , \quad (7)$$

with $\Gamma_c = 2g_0^2/\kappa$.

14. Assume first that $\kappa \gg \Gamma_c$ and that we look at a time $t \gg 1/\kappa$. Explain that you can take c_e out of the integral. You may want to use a plot to estimate the width of the integrant.
15. Show then that $c_e(t) \approx e^{-\Gamma_c t/2}$.
16. We now come back to the general case. Use Eq. (??) to calculate $\ddot{c}_e(t)$ and write a second-order differential equation on c_e . The solution for $c_e(0) = 1$ is

$$c_e(t) = e^{-\frac{\kappa t}{2}} \left[\cosh \frac{\kappa' t}{2} + \frac{\kappa}{\kappa'} \sinh \frac{\kappa' t}{2} \right] \quad (8)$$

with $\kappa' = \kappa \sqrt{1 - 2\Gamma_c/\kappa}$.

17. Assume now that $\kappa \ll \Gamma_c$. Show that you recover the single-photon Rabi oscillations discussed in Lecture 2. Remember that $\cosh(ix) = \cos x$.
18. Plot $|c_e(t)|^2$ for $\kappa \ll \Gamma_c$.
19. Interpret the different behaviors of the system as a function of Γ_c/κ .
20. The spontaneous emission rate of a two-level atom in free space is given by $\Gamma = d_{eg}^2 \omega_0^3 / (3\pi \epsilon_0 \hbar c^3)$. Calculate the so-called Purcell factor $\mathcal{F} = \Gamma_c/\Gamma$ as a function of $\lambda = 2\pi c/\omega_0$, V and the quality factor of the cavity $Q = \omega_0/\kappa$.
21. Modern single photon sources rely on quantum dots placed between Bragg mirrors. The refraction index of the semi-conductor cavity containing the dots is $n = 3.5$, and one must take the wavelength of the light in the medium in the expression of the Purcell factor. Take $\lambda = 900 \text{ nm}$ in vacuum (i.e. when the photons have left the cavity), $V \approx 10^{-19} \text{ cm}^3$, $Q = 2000$. Calculate $\mathcal{F}$.
22. Calculate the enhancement factor $\beta = \Gamma_c/(\Gamma_c + \Gamma)$. Interpret its meaning and why this configuration is interesting for a single photon source.