

Physics of Quantum Information exam (2 hours 45)

Master 2 QLMN

03 December 2021 (09:00 - 11:45)

Allowed documents : lecture notes and problem sets. Calculators allowed. It is recalled that a clean examination paper can only put the examiner in a good mood.

Aux francophones (et francographes) : vous pouvez répondre en français.

The exam consists of three independent parts :

1. General questions about the course ;
2. A discussion of two QND experiments (including calculations) ;
3. A more formal exercise.

1 General questions about the course

Answer each questions of this first part in less than 10 lines. Precise and argued answers are, however, expected.

1. How would you make the difference experimentally between a qubit state $\alpha|0\rangle + \beta|1\rangle$ and the state described by the density matrix $\rho = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$.
2. Take a two-level atom driven by a laser with Rabi frequency $\Omega \gg \Gamma$, with Γ the linewidth of the excited state. Use the stochastic wave function approach to plot the evolution of the excited state probability $\pi_e(t)$ as a function of time for a typical quantum trajectory. Indicate the relevant time scales in the plot.
3. Take an atom in a cavity. The energy loss rate of the cavity is $\kappa = 2\pi \times 1$ GHz. The atom-cavity coupling is $g = 2\pi \times 1$ MHz. What is the value of the memory time τ_c that counts in this problem ?
4. Consider two coupled systems S and E in the state

$$|\Psi\rangle_{SE} = \frac{1}{2}|0\rangle_S \otimes (|0\rangle + |1\rangle)_E + \frac{1}{2}|1\rangle_S \otimes (|0\rangle - |1\rangle)_E.$$

Is this vector state normalized ? Is this a product state ? How would you describe the state of S ? Calculate it explicitey.

2 QND measurement of photons in a cavity

We describe here two experiments realized in 2006 [Nature **446**, 297 (2007) and Nature **448**, 889 (2007)] where the QND measurement of a field in a cavity was performed using the setup presented in Lecture 3. The experiment is represented in Fig.1. It consists of a cavity C containing photons in a single cavity mode. A beam of two-level atoms (states $|g\rangle, |e\rangle$) crosses the cavity perpendicularly to its axis. The atoms are used to perform a QND measurement of the cavity field to observe quantum trajectories and to modify the state of the cavity mode. The cavity C is placed between two microwave cavities $R_{1,2}$: they are used to prepare the atoms before they enter C and to manipulate them prior to the measurement of their state.

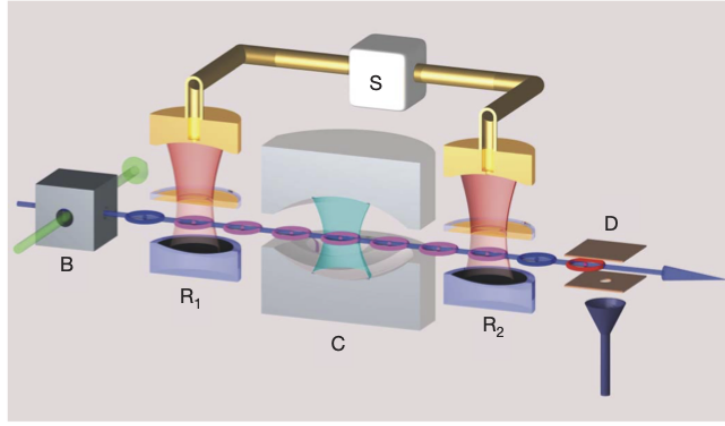


FIGURE 1 – *Experimental setup : B is the atom source, followed by a microwave cavity R_1 preparing the atoms in a given superposition of the states $|g\rangle$ and $|e\rangle$. The atoms fly through the cavity C containing n photons and then through the cavity R_2 before being detected either in g or e (D).*

2.1 Atom-field state and QND measurement

5. The microwave cavities $R_{1,2}$ are used to prepare the atoms in superposition states. They contain a *classical* field $E_0 \cos(\omega_0 t + \Phi)$, resonant with the $g - e$ transition at ω_0 , driving the atomic dipole with Rabi frequency Ω . Explain in a few lines (no calculation required) why and how they realize the rotation

$$\mathcal{R}(\Phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\Phi} \\ -e^{-i\Phi} & 1 \end{pmatrix}_{|g\rangle, |e\rangle} . \quad (1)$$

Here Φ is a phase that the experimentalists can control.

6. The atoms and the cavity C are detuned by $\delta = \omega - \omega_0$. In this dispersive regime, the cavity containing n photons imprints on the states of the atoms a phase which depends on n : $\Phi_{g,e} = \pm n\delta\varphi$ with $\delta\varphi$ the phase associated to a single photon (see Lecture 3). Take an atom initially in $|g\rangle$ before R_1 . Assume that for R_1 we

set $\Phi = 0$. Calculate the atom-field state after the cavity C when it contains n photons.

7. Represent the effect of R_1 and of the cavity C on the Bloch vector representing the atomic qubit.
8. Similarly to R_1 , the cavity R_2 realizes $\mathcal{R}(\Phi)$. Calculate the state after R_2 and show that the probability to detect the atom in g, e when the cavity contains n photons is $P(g, \Phi|n) = (1 + \cos[2n\delta\varphi - \Phi])/2$ and $P(e, \Phi|n) = (1 - \cos[2n\delta\varphi - \Phi])/2$. These two expressions can be concatenated in a single expression : $P(j, \Phi|n) = (1 + \cos[2n\delta\varphi - \Phi + j\pi])/2$ with $j = 0$ for g and $j = 1$ for e .

2.2 Birth and death of a photon

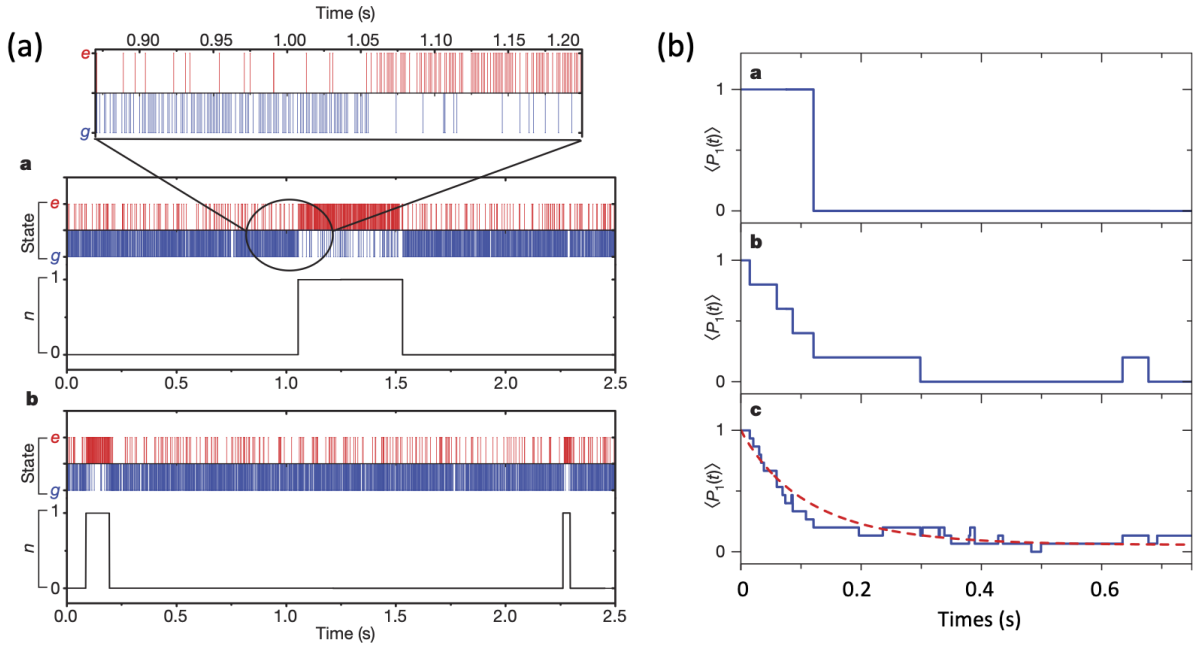


FIGURE 2 – (a) Two examples of experimental sequence corresponding each to several hundreds of atoms sent through the cavity : the atoms are either found in e or g in D , and n is the number of photons inferred from the measurements of the atomic states. (b) Probability to find a single photon in C for respectively 1, 5 and 15 trajectories.

9. In this part we want to record the quantum trajectories of photons in the cavity. Explain why you need a QND measurement to observe them.
10. We set $\Phi = 0$. What should be the value of $\delta\varphi$ to be able to associate the presence of one photon to the detection of the atom in e ?
11. Figure 2(a) shows the states of a stream of atoms sent to the cavity which is initially nearly empty. Explain the signal.
12. The temperature of the cavity C is $T = 0.8$ K. Explain why a photon can appear and calculate the mean photon number $\bar{n}$ for $\omega_0 = 2\pi \times 51$ GHz (Boltzmann and Planck constants : $k_B \simeq 1.38 \times 10^{-23}$ J/K and $h \simeq 6.63 \times 10^{-34}$ m²kg/s).

13. To prepare a single photon in the cavity in a deterministic way, an atom is initially prepared in $|e\rangle$ and sent through it. In contrast to the situation discussed above, the cavity C and the atom are resonant for this preparation. Explain how this allows to prepare a single photon (no calculation required).
14. Once the photon is prepared, the experimentalists send a stream of atoms in the cavity to perform the QND measurement described above. Figure Fig. 2(b) shows the probability $P_1(t)$ that one photon is present in the cavity for, respectively, a single trajectory and an average of 5 and 15 quantum trajectories. Explain the signal.
15. Estimate the lifetime T_c of the photon in the cavity. What is the distance travelled by a photon before it exists the cavity ? If the distance between the mirrors is 2.7 cm, how many round-trips did it correspond to ?

2.3 Collapse of a photon field under a QND measurement

16. We now assume that the cavity contains a coherent state corresponding to a Poisson distribution of the number of photons $P_0(n)$. The average is $\bar{n} \approx 4$. We set the single photon phase shift $\delta\varphi = \pi/(2q)$, where q is an integer. Plot the various probabilities $P(e, \Phi|n)$ to obtain the atom in e on a trigonometric circle for $q = 4$ and $\Phi = 0$. Explain why you would not be able to distinguish more that $2q$ photons.
17. We want to calculate the probability $P(n|e, \Phi)$ that the cavity contains n photons conditioned to the measurement of an atom in e for a particular phase Φ . Justify that :

$$P(n|e, \Phi) \propto P_0(n)P(e, \Phi|n) . \quad (2)$$

18. We now set $\Phi = p\pi/q$, with p also an integer. Explain the effect of measuring an atom in e on the distribution $P(n|e, \Phi)$.
19. Assume now that you send a stream of N atoms through C . The atom k is measured either in g ($j(k) = 0$) or e ($j(k) = 1$) for a choice of phase $\Phi(k)$. Write the probability distribution of the photon number $P_N(n)$ as a product of $P(j(k), \Phi(k)|n)$ and $P_0(n)$ (ignore the normalization constant).
20. The result of the experiment for a particular sequence of detections of atoms (j in Fig. 3 Left) associated to a particular choice i of the phase Φ between 4 values a, b, c, d is shown in Fig. 3 (Right). Describe the results and interpret them qualitatively.

<i>j</i>	1101111111110011101101111
<i>i</i>	ddcbccabcbdaadaabaddbdbc
<i>j</i>	0101001101010101010101111
<i>i</i>	dababbaacbccdadccdcbaaacc

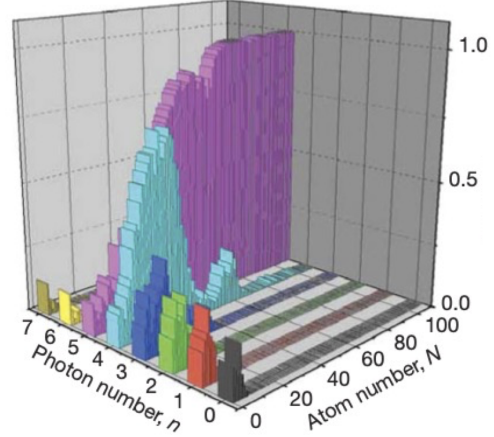


FIGURE 3 – (Left) Sequence of results of the measurement of the state j of the atoms for a particular choice of phase Φ_i taken between 4 values a, b, c, d . (Right) Evolution of the photon number distribution as a function of the number of atoms sent through the cavity.

3 Coherent versus dissipative coupling

The aim of this exercise is to compare the dynamics of a qubit subjected to two kinds of couplings, both generated by the operators $\hat{\sigma}_{\pm}$. The two qubit states $|0\rangle$ and $|1\rangle$ are assumed to have the same energy.

3.1 Coherent coupling

Consider first the coherent coupling induced by the Hamiltonian

$$\hat{H} = \frac{\hbar\Omega}{2} (\hat{\sigma}_+ + \hat{\sigma}_-), \quad (3)$$

where Ω is a real, positive number.

21. Show that the eigenstates of $\hat{H}$ are the states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and find the corresponding eigenenergies $E_{\pm}$.
22. Using the quantum Liouville equation, $\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}]$, write the equations for the populations $\rho_{\pm,\pm}$ and the coherences $\rho_{\pm,\mp}$, in the $|\pm\rangle$ basis. Solve them as a function of $\rho_{+,+}(0)$, $\rho_{-,-}(0)$, $\rho_{+,-}(0)$, and $\rho_{-,+}(0)$.
23. Find the general expression of the population $\rho_{1,1}(t)$ as a function of $\rho_{+,+}(0)$, $\rho_{-,-}(0)$, $\rho_{+,-}(0)$, and $\rho_{-,+}(0)$.
24. Assume that the qubit is prepared in the pure state $|0\rangle$. What are the expressions of $\rho_{+,+}(0)$, $\rho_{-,-}(0)$, $\rho_{+,-}(0)$, and $\rho_{-,+}(0)$?
25. Plot $\rho_{1,1}(t)$ versus t , indicating the characteristic time(s). What well-known phenomenon do we recover?

3.2 Dissipative coupling

Consider now a dissipative coupling induced by the Lindblad operators $\sqrt{\Gamma_+}\hat{\sigma}_+$ and $\sqrt{\Gamma_-}\hat{\sigma}_-$.

26. What do these Lindblad operators represent physically? What is the physical meaning of the quantities $\Gamma_{\pm}$?
27. Write the Lindblad formula and find the equations for the time evolution of the populations $d\rho_{0,0}/dt$ and $d\rho_{1,1}/dt$. Note that no coherent coupling is applied.
28. Find the stationary values $\rho_{0,0}^{\text{stat}}$ and $\rho_{1,1}^{\text{stat}}$.
29. Solve the equations found at question 27.
30. Assume as above that the qubit is prepared in the pure state $|0\rangle$. Plot $\rho_{1,1}(t)$ versus t , indicating the characteristic time(s) and the stationary population.
31. Compare to what you found at question 25. What is the physical origine of the difference?