Lecture 2 - Exercise correction

Exercise 1. *Compute the speed of a Cesium 133 atom in a 300K environment. Considering using a laser tuned to be on resonance with the D_2 line, compute the typical number of emission-absorption cycles this atom must undergo to reach a temperature of $100\,\mu K$. Assuming that Cesium atoms travel along a well defined direction, compute the typical distance which is required to cool and trap a Cesium atom. Deduce the typical size of a vacuum system used in a neutral atom quantum computer.*

The Cesium most-probable speed is $v(300K) = \sqrt{2kT/m} \simeq 200$m/s. At each absorption, Cesium looses a speed $v_r = \hbar k/m$ where $k = 2\pi/\lambda$ with $\lambda = 852$ nm. We obtain $v_r \simeq 3.5 \times 10^{-3}$m/s. The most probable speed at $100\,\mu$K is $v(100\mu K) \simeq 10$cm/s. We therefore obtain a typical number of cycles $N \simeq 6 \times 10^4$ to reach this speed. A Cesium atom can scatter at most about $\Gamma/2$ photons on the D2 line, with $\Gamma \simeq 2\pi \times 5.2$MHz, meaning that the scattering rate is roughly $R \sim 1.6 \times 10^7$s^{-1}. This means that cooling Cesium takes a time $t \sim 4 \times 10^{-3}$s. Since the exterted force on the atom is constant, its mean speed during cooling is about 100m/s, meaning that the typical distance which is required to slow Cesium is $d \sim 0.4$m. The typical linear size of the vaccuum system of a neutral atom quantum computer is therefore in the range of tens of centimeter.

Exercise 2. *Assuming a Be^+ ion in a Paul trap as described above with $V_0 = 100$ V and $z_c = 1$ mm, compute the minimal value of ω_{rf} which satisfies the condition on the stability region described above.*

We obtain $\omega_{\rm rf} \geq 4.8 \times 10^7$ rad/s and subsequently $f_{\rm rf} = \omega_{\rm rf}/2\pi \simeq 7.7$MHz, meaning that stable trapping of ions requires MHz-scale frequency variations of the trapping potential.

Exercise 3. *Considering rubidium 87, derive the impact of black-body radiation given above ($T_1^{BBR} \sim 10^{-12}$ s) at 300K using the thermal photon occupation formula of a 300K environment.*

At 300K, the thermal occupation number at the relevant frequency here ($f_0 = 6.8$GHz) is $\bar{n} = 1/(e^{\hbar 2\pi f_0/k_B T} - 1) \sim 900 \sim 10^3$. The stimulated rate induced by BBR is thus: $\Gamma_{\rm BBR} = \bar{n} T_1 \sim 10^{-9}$s^{-1}, meaning that we expect one event every 1 billion seconds: negligible.

Exercise 4. *Considering rubidium 87, compute the Raman scattering rate for light at 820 nm and assuming a trap depth of 1 mK. Also compute the laser power required to trap an atom, assuming that the optical tweezers has a Gaussian profile with a radius waist of $1\,\mu m$. Perform the same computations (Raman scattering rate and required laser power) now assuming a wavelength of 850nm. What conclusion do you reach from*

these calculations? Can you provide an advice on the tweezers wavelength one should use to build a neutral atom quantum computer?

For a wavelength of 820nm, $\Delta \simeq 2\pi \times 20$THz, the linewidth of the D2 line is $\Gamma \simeq 2\pi \times 6$ MHz and $\Delta_{\text{FS}} = 2\pi \times 7$THz. The Raman scattering rate is $\Gamma_R \simeq \Gamma U \Delta_{\text{FS}}^2/3\hbar\Delta^3 \simeq 0.8\text{s}^{-1}$, meaning that $T_1 \simeq 1$s for 820nm light. For 850nm light, we find $\Gamma_R \simeq 0.16$Hz, meaning $T_1 \simeq 6$s.

The laser power P which is required to produce a trap with $U = 1$mK depth on the D2 line atomic transition frequency w_0 (780nm) and with natural linewidht Γ, considering a waist $w = 1\,\mu$m, is: $P = U w^2 w_0^3 \Delta/(3c^2\Gamma) \simeq 2.2$mW for 820nm tweezers and $P \simeq 3.8$mW for 850nm tweezers. There is therefore a trade-off between available laser power and qubit quality: the higher the wavelength, the higher the T_1, but the higher is the cost in terms of laser power to trap the atoms. Assuming a fixed amount of laser power, one should aim for the highest wavelength which still allows one to trap the required number of qubits for a specific application.

Exercise 5. *Considering rubidium 87, compute the energy shift that is applied to the ground states due to a static electric field of 1 V/cm (compare with the situation of 0 electric field). Perform the same for a magnetic field of 1 Gauss. Out of the 8 ground states of Rubidium, find the best qubit based on these calculations.*

An electric field $E = 1$V/cm applies a Stark shift $\Delta E_{\text{Stark}} = -0.5\alpha E^2$ with α the polarizability of the considered states. The polarizability of the ground states is essentially equal for all states and is $\alpha \simeq 47 \times 10^{-24}\,\text{cm}^3$, giving $\Delta E_{\text{Stark}} \simeq 0.036$ Hz.

A magnetic field of strength $B = 1$G applies a Zeeman shift $\Delta E_Z = g_F \mu_B m_F B$ where g_F is the Lande factor with values $\pm 1/2$ for $F = 2$ and $F = 1$ respectively, $\mu_B \simeq 9.3 \times 10^{-24}\,\text{J/T}$ is the Bohr's magneton, and m_F is the magnetic quantum number, with values $-2, -1, 0, 1, 2$ depending on the considered states. We obtain:

- $F = 1, m_F = -1 \to \Delta E_Z = 0.7$ MHz

- $F = 1, m_F = 0 \to \Delta E_Z = 0$ MHz

- $F = 1, m_F = 1 \to \Delta E_Z = -0.7$ MHz

- $F = 2, m_F = -2 \to \Delta E_Z = -1.4$ MHz

- $F = 2, m_F = -1 \to \Delta E_Z = -0.7$ MHz

- $F = 2, m_F = 0 \to \Delta E_Z = 0$ MHz

- $F = 2, m_F = 1 \to \Delta E_Z = 0.7$ MHz

- $F = 2, m_F = 2 \to \Delta E_Z = 1.4$ MHz

There are two states for which there is no linear Zeeman effect, which thus consitutes the best states for defining a qubit: $|5S_{1/2}, F = 1, m_F = 0\rangle$ and $|5S_{1/2}, F = 2, m_F = 0\rangle$.

Exercise 6. *Find the expression of κ.*

$$\kappa = \tan^{-1}((12\beta^2\bar{\Delta}_{\mathrm{LS}}t - \bar{\Delta}_{\mathrm{LS}}^3 t^3)/(8\beta^3 - 6\beta\bar{\Delta}_{\mathrm{lS}}^2 t^2))$$

Exercise 7. *Demonstrate that $P_0 = 1$ in the echo sequence described above.*

A π rotation about y flips z: $R_y(\pi/2)R_z(\theta)R_y(\pi)^\dagger = R_z(-\theta)$. Since $R_y(\pi)^\dagger = R_y(-\pi) = R_y(\pi)$ (up to a global phase), we can rewrite: $R_y(\pi)R_z(\theta) = R_z(-\theta)R_y(\pi)$. Therefore $R_y(\pi/2)R_z(\delta t)R_y(\pi)R_z(\delta t)R_y(\pi/2) = R_y(\pi/2)R_y(\pi)R_y(\pi/2) = R_y(2\pi)$, hence a 2π rotation, and therefore $P_0 = 1$.

Exercise 8. *Show on a Ramsey experiment that changing the phase of the second pulse by a quantity ϕ is equivalent to applying $R_z(\phi)$ during the Ramsey experiment.*

We have $H = \frac{\hbar\Omega}{2}(\cos(\phi)X + \sin(\phi)Y)$, and we assume a Ramsey experiment U_0 with $\phi = 0$ on the first and second pulses: $U_0 = R_x(\pi/2)R_z(\delta t)R_x(\pi/2)$. Now considering the case where the second pulse is performed with phase ϕ, we perform the unitary $R_\phi = R_z(\phi)R_x(\pi/2)R_z(-\phi)$. We therefore obtain $U = R_x(\pi/2)R_z(\delta t)R_\phi(\pi/2) = R_x(\pi/2)R_z(\delta t)R_z(\phi)R_x(\pi/2)R_z(-\phi)$. Since a Ramsey experiment measures population in the computational basis, any phase rotation before or after x rotations do not impact the population. Therefore, $R_z(-\phi)$ does not have any incidence on the Ramsey experiment, and thus we obtain that changing the phase of the second pulse by ϕ is equivalent to applying $R_z(\phi)$.

Exercise 9. *Show how to implement a local $R_y(\theta)$ rotation using global $\bar{R}_x$ rotations and local R_z rotations.*

$$R_y(\theta) = \bar{R}_x(\pi/2)R_z(\theta)\bar{R}_x(-\pi/2)$$

Exercise 10. *Assuming that $\Omega_1 = \Omega_2$, demonstrate the formula giving the amount of scattering during a π pulse.*

The two-photon Rabi frequency is $\Omega = \Omega_1\Omega_2/(2\Delta) = \Omega_1^2/(2\Delta)$. The effective scattering rate from the intermediate state is $\Gamma_{\mathrm{eff}} = \gamma(\Omega_1^2 + \Omega_2^2)/(4\Delta^2) = \gamma\Omega_1^2/(2\Delta^2)$ with γ the scattering rate from the intermediate state. We thus obtain $\Gamma_{\mathrm{eff}} = \gamma\Omega/\Delta$. Following Rabi's formula, the excitation duration t for performing a π pulse is $t = \pi/\Omega$, meaning that $P_{\mathrm{scat}} = \Gamma_{\mathrm{eff}}t = \pi\gamma/\Delta$.

Exercise 11. *Assuming Rubidium 87, compute the typical one-photon Rabi frequency using a laser on resonance on the D_1 line (795 nm) using a laser with a power of 1 mW with a Gaussian profile and a radius waist of 100 μm. Assuming a second laser identical to the first one and set to perform the Raman transition, and a measured $T_1 = 10$ ms for both qubit states, what is the optimal value of Δ which maximizes the single qubit gate fidelity, and what is the expected fidelity?*

The Rabi frequency Ω_1 is given by $\Omega_1 = \gamma\sqrt{I/(2I_{\text{sat}})}$ with $\gamma \simeq 2\pi \times 5.8$ MHz the linewidth of $5P_{1/2}$, and $I_{\text{sat}} \simeq 3.6\text{mW/cm}^2$ the saturation intensity of the D1 line. The laser intensity is $I = 2P/(\pi w^2) = 6 \times 10^{-3}\text{W/m}$, giving $\Omega_1 \simeq 2\pi \times 170$ MHz (we here do not consider Clebsch-Gordan coefficients for simplicity).

We assume performing a π pulse, meaning that the excitation time is $t = \pi/\Omega = 2\pi\Delta/\Omega_1^2$. We need to consider two decoherence mechanisms:

- The T_1-induced decoherence with probability $P_T = 1 - e^{-t/T_1} = 1 - e^{-2\pi\Delta/T_1\Omega_1^2}$.

- The decay from the intermediate state with probability $P_{\text{scat}} = 1 - e^{-\Gamma_{\text{eff}}t} = 1 - e^{-t\gamma\Omega_1^2/(2\Delta^2)} = 1 - e^{-\pi\gamma/\Delta}$. Interestingly, this decoherence channel does not depend on Ω_1: the gate duration is $t \propto 1/\Omega_1^2$, and the decoherence rate is $\Gamma_{\text{eff}} \propto \Omega_1^2$.

The total decoherence probability is the multiplication of both effects: $P_{\text{err}} = 1 - e^{-2\pi\Delta/T_1\Omega_1^2 - \pi\gamma/\Delta}$. The optimum is found when $dP/d\Delta = 0$, meaning $2\pi/(T_1\Omega_1^2) = \pi\gamma/\Delta^2$, and thus for $\Delta = \sqrt{\gamma T_1\Omega_1^2/2} \simeq 2\pi \times 72\text{GHz}$. This gives a fidelity $F = 1 - P_{\text{err}} \simeq 99.95\%$.

Exercise 12. *Assuming an energy level repartition following the hydrogen structure, find the scaling $U_{VdW} \propto n^{11}$.*

As seen in the lecture, $U_{\text{VdW}} \propto d^4/\Delta$. Since the electron's orbit scales as n^2, the dipole size also scales as $d \propto n^2$. For hydrogen, the states energies are $E \propto 1/n^2$, meaning that $\Delta \sim dE/dn \propto 1/n^3$. Plugging the scalings, we get $U_{\text{VdW}} \propto n^{11}$.

Exercise 13. *Demonstrate that in the regime $\Omega_r/(2\pi) \ll U_{VdW}/h$, the two atom Hamiltonian applied to $|00\rangle$ has a coupling strength of $\sqrt{2}\Omega_r$ to the $|W\rangle$ state.*

We apply H^2_{ryd} to $|00\rangle$. Since $\Omega_r/(2\pi) \ll U_{\text{VdW}}/h$, $|rr\rangle$ is far-detuned and therefore the system can only couple to $|0r\rangle$ and $|r0\rangle$. Assuming $\phi_r = 0$, we obtain $H^2_{\text{ryd}}|00\rangle = \frac{\hbar\Omega_r}{2}(|0r\rangle + |r0\rangle) = \frac{\hbar(\sqrt{2}\Omega_r)}{2}|W\rangle$, meaning that the coupling to $|W\rangle$ is $\sqrt{2}\Omega_r$.

Exercise 14. *Assuming Yb^+, a laser at 355 nm and $\omega_z = 2\pi \times 5$ MHz, compute the typical Lamb-Dicke parameter value.*

$\eta = \frac{2\pi}{\lambda}\sqrt{\frac{\hbar}{2m\omega_z}} \simeq 0.14$.

Exercise 15. *Assuming the neutral atom platform capabilities in terms of single- and two-qubit gates, describe a circuit which allows to implement a CNOT gate. Describe a circuit that allows to implement a Toffoli gate (described in Lecture 1).*

From the lecture, neutral atom: abitrary single-qubit gates and CZ gate. A CNOT gate is composed of $CNOT_{c\to t} = (I \otimes H)CZ(I \otimes H)$, meaning that the protocol is:

- Apply Hadamard on the target

- Apply CZ between control and target

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- Apply Hadamard on the target

For the Toffoli, we denote a, b the controls qubit, and c the target qubit. We have $CCNOT_{a,b\to c} = (I \otimes I \otimes H)CCZ(I \otimes I \otimes H)$. Defining the T-gate $T = R_z(\pi/4)$, the CCZ gate is performed by doing:

1. a T-gate on the 3 qubits (T_a, T_b, T_c)

2. $CZ_{a,b}$ followed by $T_b^\dagger$.

3. $CZ_{b,c}$ followed by $T_c^\dagger$.

4. $CZ_{a,c}$ followed by T_c.

5. $CZ_{b,c}$ followed by $T_c^\dagger$.

6. $CZ_{a,c}$ followed by T_c.

7. $CZ_{a,b}$ followed by T_b.

which starts to be a lot of gates.

Exercise 16. *Same exercise but now considering the trapped ion platform.*

From the lecture, trapped ions: arbitrary single-qubit gates and $R_{YY}(\pi/2)$ gate. $CNOT_{c\to t} = R_x^t(-\pi/2)R_y^c(\pi/2)R_{YY}(\pi/2)R_y^c(-\pi/2)R_x^t(\pi/2)$.
The CCNOT gate can be built upon the CNOT gate via:

- H_c

- $CNOT_{b\to c}$ followed by $T_c^\dagger$

- $CNOT_{a\to c}$ followed by T_c

- $CNOT_{b\to c}$ followed by $T_c^\dagger$

- $CNOT_{a\to c}$ followed by T_b

- T_b, T_c and H_c

where each CNOT has been decomposed into the native gates above.