Quantum Computing Lecture 1
Solutions to the Problems

Exo 1

XOR

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

NAND

x	y	$\overline{x \cdot y}$
0	0	1
0	1	1
1	0	1
1	1	0

Ex 2

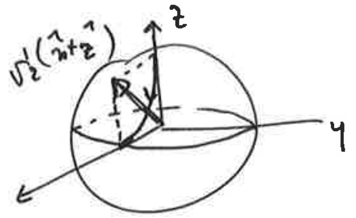
- $R_{\frac{\hat{x}+\hat{z}}{\sqrt{2}}}(\pi) = -i \sin\frac{\pi}{2} \; \frac{\hat{x}+\hat{z}}{\sqrt{2}} = -\frac{i}{\sqrt{2}}(\hat{x}+\hat{z}) = -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -iH$

Hence $\hat{H} = +i\, R_{\frac{\hat{x}+\hat{z}}{\sqrt{2}}}(\pi)$

- $R_y(\pi/2) R_z(\pi) = -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -i\hat{H}$

Hence $\hat{H} = i\, R_y(\pi/2)\, R_z(\pi/2)$

Note: the common error is to think that $\hat{H} = R_y(\pi/2)$. This is not!



Exo 3

$$\hat{U} = e^{i\alpha} R_z(\beta)\, R_y(\gamma)\, R_x(\delta)$$

$$\hat{U} = e^{i\alpha}\begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}\begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\delta}{2} & -i\sin\frac{\delta}{2} \\ i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}$$

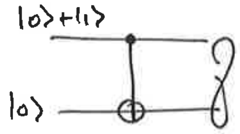
$$= \dots = \begin{pmatrix} e^{i(\alpha-\beta/2-\frac{\gamma}{2})}\cos\frac{\gamma}{2} & -e^{i(\alpha-\frac{\beta}{2}+\frac{\delta}{2})}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\frac{\beta}{2}-\frac{\delta}{2})}\sin\frac{\gamma}{2} & e^{i(\alpha+\frac{\beta}{2}+\frac{\gamma}{2})}\cos\frac{\gamma}{2} \end{pmatrix}$$

This is the general form of a unitary matrix (2×2):

$$\hat{U} = \begin{pmatrix} a & b \\ -e^{i\varphi}b^* & e^{i\varphi}a^* \end{pmatrix}$$ And $\det U = e^{i\varphi}$ (See Wikipedia english)

Exo 4

CNOT take



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

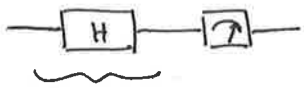
ϕ gate

$|\psi_{out}\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$

which is entangled

(check it by looking at a decomposition $|\psi_{out}\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$)

Exo 5



Rotate the state before performing the meas[t] in the $|x\rangle$ basis:

Exo 6

Start from a 2 qbit state $|0\rangle|0\rangle$ and apply a sequence of gates $\hat{S}$, $\hat{H}$ and CNOT : you will only generate states with amplitudes $\left(\pm\frac{i}{\sqrt{2}}\right)^n$ or $\left(\frac{1}{\sqrt{2}}\right)^n$: they are in the equatorial plane of the Bloch sphere. Adding the $\hat{T}$ gate allows you out of this plane and gives amplitudes $\left(\frac{1+i}{\sqrt{2}}\right)^n$ corresponding to states anywhere on the sphere.

Exo 7

$$e^{iAt/n} \simeq 1 + i\frac{At}{n} + o(1/n^2)$$

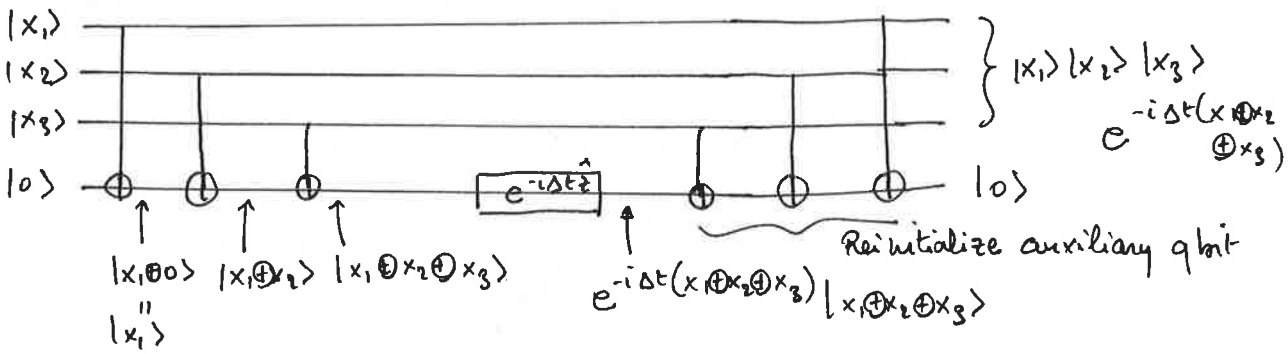
$$e^{iBt/n} \simeq 1 + i\,Bt/n + o(1/n^2)$$

$$\Rightarrow \quad e^{iAt/n}\, e^{iBt/n} \simeq 1 + i\,(A+B)t/n + o(1/n^2)$$

$$\Rightarrow \quad \left(e^{iAt/n}\, e^{iBt/n}\right)^n \simeq \left(1 + i\,(A+B)t/n\right)^n$$

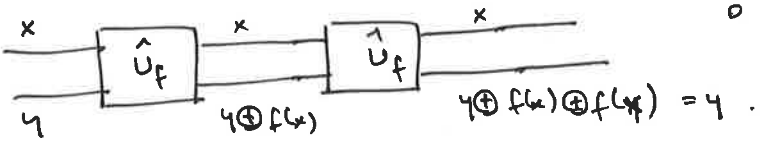
But $\lim_{n\to\infty}\left(1 + i\,\hat{O}/n\right)^n = e^{i\hat{O}} \Rightarrow \lim_{n\to\infty}\left(e^{iAt/n}\, e^{iBt/n}\right)^n = e^{i(A+B)t}$.

Exo 8

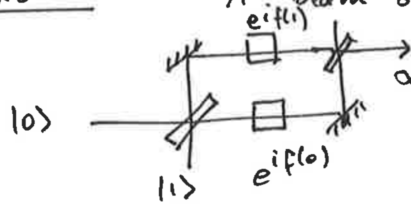


Exo 9 Apply U_f twice:

$$U_f |xy\rangle \Rightarrow |x, y \oplus f(x)\rangle \xrightarrow{U_f} |x, y \oplus \underbrace{f(x) \oplus f(x)}_{=0}\rangle = |x, y\rangle$$



Exo 10 A Beam splitter in optics implements the Hadamard transformation



$$\propto |0\rangle \left[e^{if(0)} + e^{if(1)}\right] + |1\rangle \left[e^{if(0)} - e^{if(1)}\right]$$

Exo 11 $L = 2048$

classical algorithm $\# \sim e^{1.9 L^{1/3} (\ln L)^{2/3}}$ Number of operation

$1.9 L^{1/3} \ln L^{2/3} \approx 93 \Rightarrow \# \simeq 10^{31}$ @ 1 GHz $\Rightarrow 10^{31} \times 10^{-9}$ sec
10^{22} sec.

One year $\simeq 3 \times 10^7$ sec $\Rightarrow t \simeq 10^{15}$ years...

Shor's algorithm $\# \sim L^3 \Rightarrow (2048)^3 \simeq 8 \times 10^9$ operations

If the Q Computer operates at GHz clock rate $\Rightarrow \simeq 10$ sec.

Exo 12 (No cloning theorem)

Take 2 states $|\phi\rangle$ and $|\psi\rangle$ totally arbitrary.

Cloning means $\exists \hat{U}$ / $U|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle$ with $U^\dagger U = 1$
$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$

$$\Rightarrow \langle\psi|\langle\psi||\phi\rangle|\phi\rangle = \langle\psi|\langle 0| U^\dagger U |\phi\rangle|0\rangle = \langle\psi|\phi\rangle$$

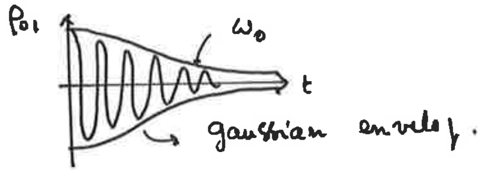
$$\Rightarrow |\langle\psi|\phi\rangle|^2 = \langle\psi|\phi\rangle \Rightarrow \langle\psi|\phi\rangle = 0 \text{ or } 1$$

Hence only possible if $|\phi\rangle$ and $|\psi\rangle$ are identical or orthogonal
Can not be arbitrary $\Rightarrow$ violates hypothesis hence impossible.

Exo 13

$$\langle \rho_{01}(t)\rangle \propto \int e^{-i\omega_i t} e^{-(\omega_i - \omega_0)^2/2\Delta\omega^2} d\omega_i \sim e^{-\Delta\omega^2 t^2/2} e^{-i\omega_0 t}$$

as $\int e^{ixu} e^{-\sigma^2 u^2} du = e^{-x^2/4\sigma^2}$



Exo 14 when no dephasing, the Ramsey sequence gives:

$$|0\rangle \underset{①}{\xrightarrow{R_y(\pi/2)}} \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \xrightarrow{②} \frac{1}{\sqrt{2}}(|0\rangle+e^{-i\delta t}|1\rangle) \underset{③}{\xrightarrow{R_y(\pi/2)}} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle)e^{-i\delta t}$$

$$\Rightarrow \quad \frac{1}{2}(1-e^{-i\delta t})|0\rangle + \frac{1}{2}(1+e^{i\delta t})|1\rangle = |\psi_f\rangle$$

Hence $P_0(T) = \sin^2\frac{\delta T}{2}$

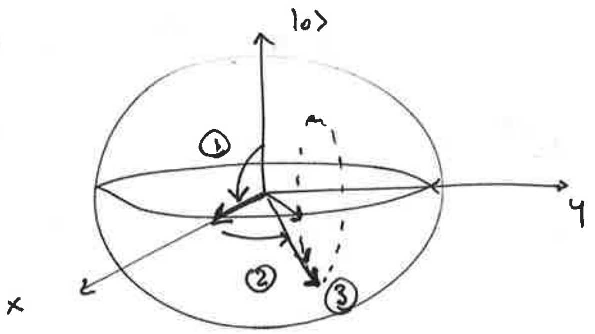
If you take $p(\delta) = e^{-\frac{(\delta-\delta_0)^2}{2\Delta^2}}\frac{1}{\sqrt{2\pi}\Delta}$, centred on δ_0

$$\Rightarrow \quad \langle P_0(T)\rangle = \int \sin^2\frac{\delta t}{2}\frac{1}{\sqrt{2\pi}\Delta}e^{-\frac{(\delta-\delta_0)^2}{2\Delta^2}}d\delta$$

$$= \frac{1}{2} - \frac{1}{2}\int \cos\delta t \, \frac{e^{-\frac{(\delta-\delta_0)^2}{2\Delta^2}}}{\sqrt{2\pi}\Delta}d\delta$$

$$= \frac{1}{2} - \frac{1}{2}e^{-T^2\Delta^2/2}\cos\delta_0 T \Rightarrow \quad T_2 \sim \frac{1}{\Delta}$$

Exo 15



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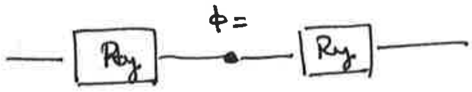
Exo 16



Interferometrically