Physics of Quantum Information

Lecture 9 – Recap of the Lectures

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Lecture 1 : Basic Laws of Quantum Physics

Quantum physics is based on a few number of basic principles. While most of the lectures pointed out that this formulation is insufficient to (almost) any practical purpose, all we did directly follows from these principles (none has been questioned, no additional principle).

Isolated systems

State represented by a ket (vector state) in a Hilbert space

Quantum superposition and entanglement

Superposition of states, $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$

Entanglement, $|\Psi\rangle=\alpha|0\rangle_A|0\rangle_B+\beta|1\rangle_A|1\rangle_B$

Evolution (out of measurements)

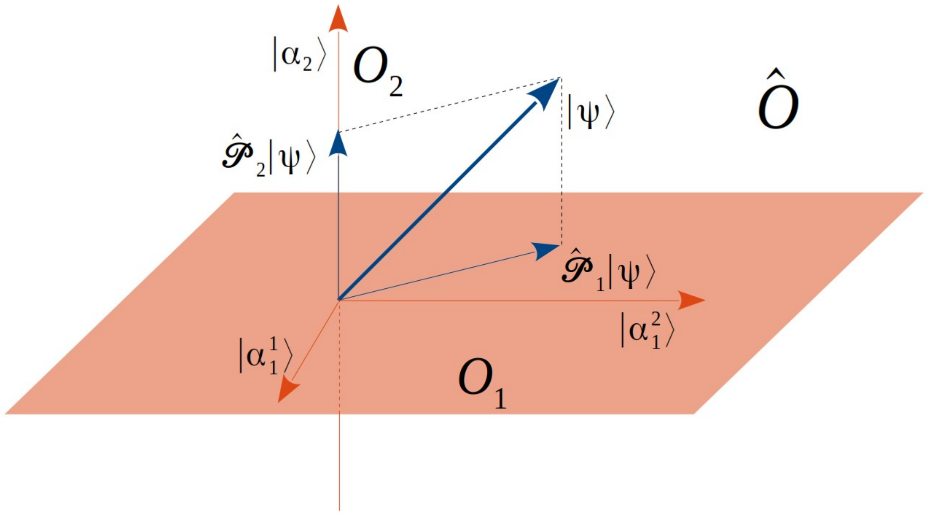
Schrödinger equation : $i\hbar\dfrac{d|\psi(t)\rangle}{dt}=\hat{H}(t)|\psi(t)\rangle$

Evolution operator : $|\psi(t)\rangle=\hat{U}(t)|\psi(0)\rangle$

→ Unitary evolution (conserves norm and orthogonality, reversible)

Projective measurements

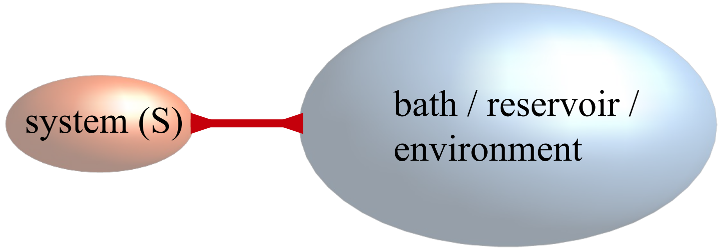
$$\hat{\mathcal{O}}=\sum_j \mathcal{O}_j\hat{\mathcal{P}}_j \quad \text{(spectral decomposition)}$$



Lecture 1+ : Reformulating the Basic Laws of Quantum Physics

The basic principles, however, only apply to *isolated* systems. In spite of dramatic experimental progress, a quantum system is never perfectly isolated (coupling to even vacuum, manipulation, and measurement devices).

The basic principles only apply to the system coupled to its environment (bipartite system). This may imply a huge amount of entanglement



$$|\Psi\rangle_{\mathrm{S\otimes E}}=\sum_{n,m} c_{n,m}|n\rangle_{\mathrm{S}}\otimes|m\rangle_{\mathrm{E}}$$

This state lives in a huge Hilbert space with dimension $\dim(\mathscr{E}_S)\times\dim(\mathscr{E}_E)\gg\dim(\mathscr{E}_S)^2$.

→ If, however, we focus on *S*, we may disregard *E* but this implies a reformulation of quantum physics (from the point of view of *S*).

State of *S* represented by a density matrix

$$|\Psi\rangle_{\mathrm{S\otimes E}}=\sum_m c_m|\psi_m\rangle_{\mathrm{S}}\otimes|\chi_m\rangle_{\mathrm{E}} \quad\rightarrow\quad \hat{\rho}_{\mathrm{S}}=\sum_m |c_m|^2|\psi_m\rangle\langle\psi_m|$$

normalized

orthonormal basis

Lecture 1+ : Reformulating the Basic Laws of Quantum Physics

Open systems

The density matrix of *S* can always be diagonalized

$$\hat{\rho}_S=\sum_n \Pi_n|n\rangle\langle n|$$

classical-like randomness

quantum randomness
(upon measurement)

Quantum superpositions

Superposition of basis states appear in coherences, $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle \quad\rightarrow\quad \hat{\rho}_S=\begin{bmatrix}|\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2\end{bmatrix}$

Evolution (out of measurements)

Master equation (counterpart of the Schrödinger equation) : $\frac{d\hat{\rho}}{dt}=\frac{1}{i\hbar}[\hat{H}(t),\hat{\rho}(t)]+\mathscr{L}'[\hat{\rho}(t)]$

Quantum map/operation/channel : $\hat{\rho}(t)=\sum_\alpha \hat{M}_\alpha\hat{\rho}(0)\hat{M}_\alpha^\dagger$

→ Nonunitary evolution (conserves norm, $\mathrm{Tr}(\hat{\rho})=1$, but not the purity, $\mathrm{Tr}(\hat{\rho}^2)$; in general irreversible)

Average of an observable $\langle\mathcal{O}\rangle=\mathrm{Tr}(\hat{\rho}\,\hat{\mathcal{O}})$

Lecture 1+ : Reformulating the Basic Laws of Quantum Physics

Bloch sphere

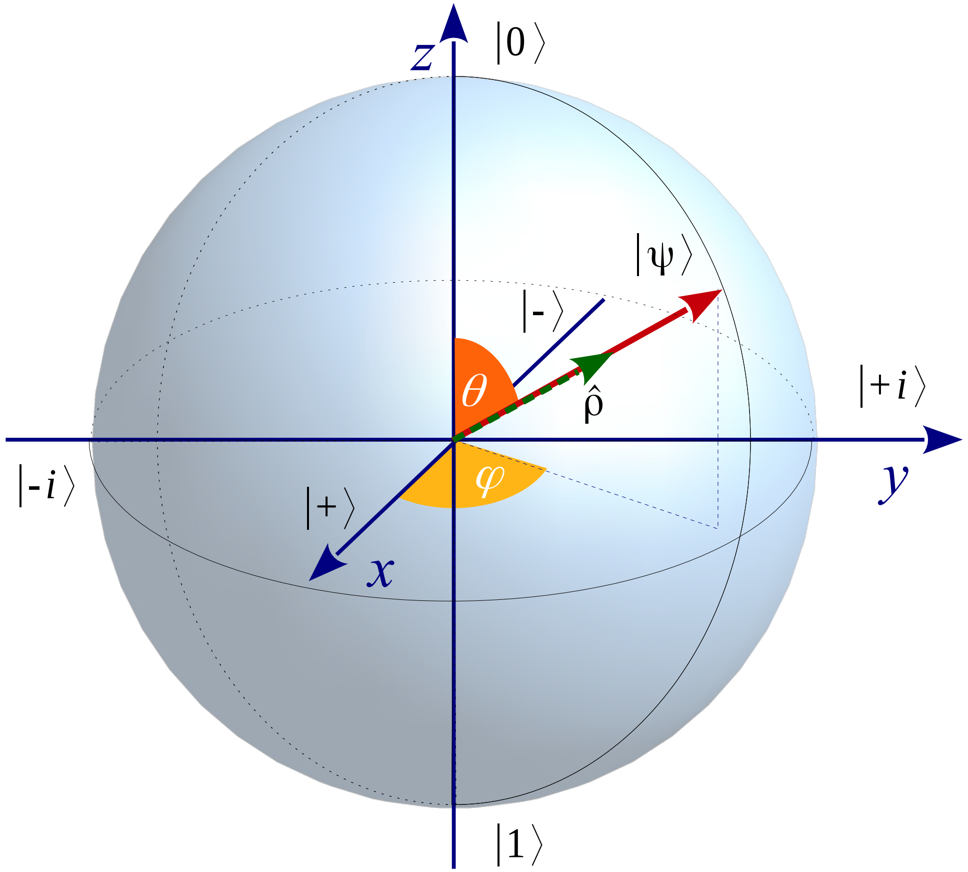
Any state of a qubit may be represented as a vector on/in the Bloch sphere, $\vec{u}=\langle\hat{\vec{\sigma}}\rangle$

Pure state :

$$|\psi\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)e^{i\varphi}|1\rangle$$

Bloch vector on the sphere, $|\vec{u}|=1$

Two orthogonal vector states (kets) are represented by opposite Bloch vectors



Mixed state :

$$\vec{u}=\mathrm{Tr}(\hat{\rho}\hat{\vec{\sigma}})$$

Bloch vector inside the sphere, $|\vec{u}|\leqslant 1$

Purity : $\mathrm{Tr}(\hat{\rho}^2)=\dfrac{1+|\vec{u}|^2}{2}$

Any single-qubit unitary operation (gate) is represented by a rotation on the Bloch sphere ($|\vec{u}|$ is conserved). For a time-independent Hamiltonian, it yields Rabi oscillations.

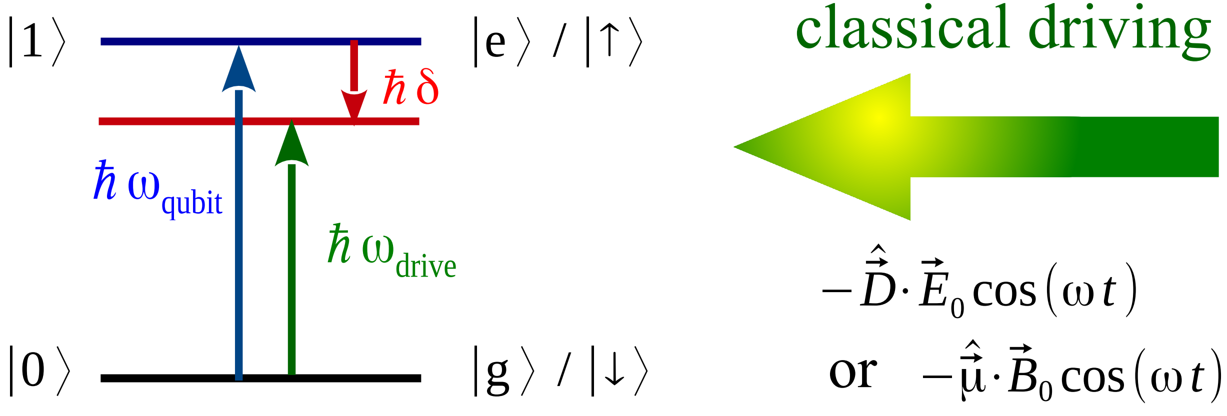
Lecture 2 : Physical Implementation and Manipulation of Qubits

Qubits can be realized by any two-state system. Driving by a classical oscillating field close to resonance allows us to perform single-qubit operations (Rabi oscillations). Coupling to quantized harmonic oscillators yields a universal Hamiltonian, aka the Jaynes-Cummings model.

Implementation and single-qubit operations

Physical qubits

- atom with 2 « priviledged » states
- particle in a two-well potential
- photons in two modes (**k**, *s*)
- real spins



single-mode
classical driving

$-\hat{\vec{D}}\cdot\vec{E}_0\cos(\omega t)$

or $-\hat{\vec{\mu}}\cdot\vec{B}_0\cos(\omega t)$ or ...

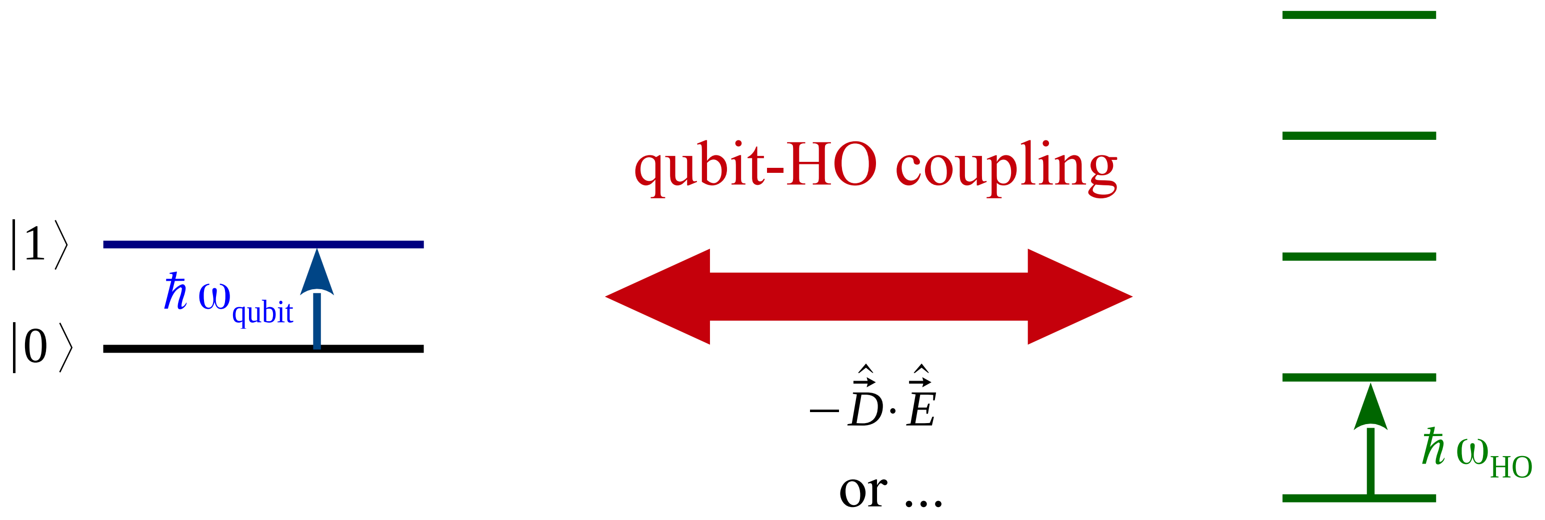
In the rotating frame and within the quasi-resonant approximation (aka RWA), it yields the effective Hamiltonian

$$\widetilde{H}_{\text{eff}} = -\frac{\hbar\delta}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$$

→ Equivalent to a spin in a controlled magnetic field ; May implement any rotation on the Bloch sphere

Lecture 2 : Physical Implementation and Manipulation of Qubits

Coupling the qubit to a quantized mode



- Atom – radiation field in vaccum
- Atom – cavity mode
- Ion – vibration mode
- Superconducting circuit – wave guide

It yields the Jaynes-Cummings Hamiltonian

$$\hat{H}_{\text{eff}} = -\frac{\hbar\,\omega_{\text{qubit}}}{2}\hat{\sigma}_z + \hbar\,\omega_{\text{HO}}\left(\hat{a}^\dagger\hat{a}+1/2\right) + \hbar\,\Omega\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a}+\hat{a}^\dagger\right)$$

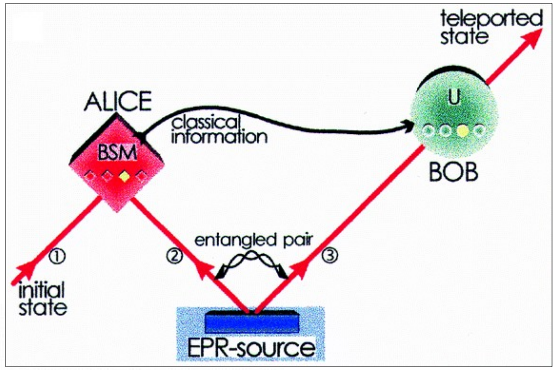
$$\simeq -\frac{\hbar\,\omega_{\text{qubit}}}{2}\hat{\sigma}_z + \hbar\,\omega_{\text{HO}}\left(\hat{a}^\dagger\hat{a}+1/2\right) + \hbar\,\Omega\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right)$$

- For the h.o. in a strong coherent state, one recovers the previous case
- For the h.o. in a number state, one find a 2-qubit coupling, $|\text{e}\rangle\otimes|n\rangle \quad \leftrightarrow \quad |\text{g}\rangle\otimes|n+1\rangle$
- For a qubit coupled to a continuum of modes (continuum of h.o.), one finds spontaneous emission

Lecture 3 : Quantum Entanglement I

Although entanglement is ubiquitous, it is very difficult to demonstrate it unambiguously (sublte nonclassical correlation). The demonstration of the violation of Bell’s inequalities has shown that quantum physics is nonlocal and closed the Bohr-Einstein debate.

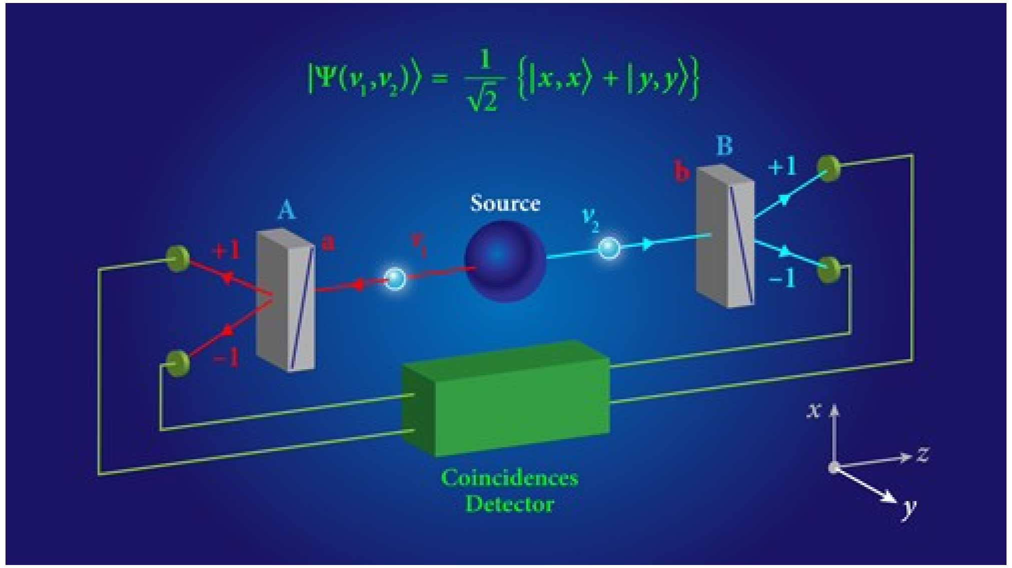
Moreover, entanglement is a resource to perform tasks that are impossible at the classical level.

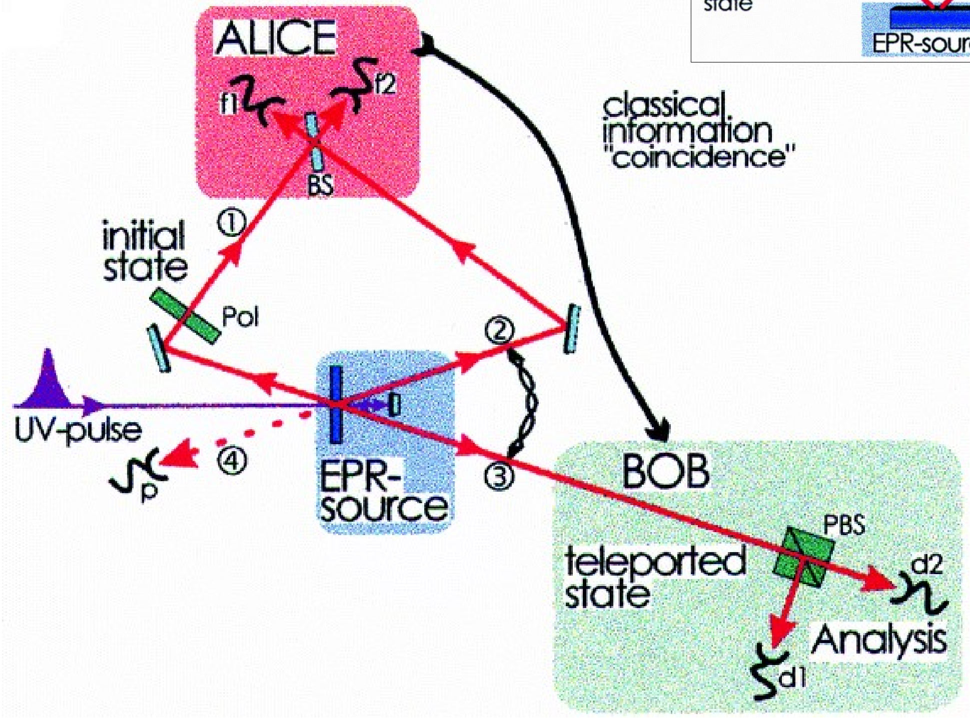


Quantum teleportation

Violation of Bell’s inequalities

Review by A. Aspect, Physics **8**, 123 (2015)





Lecture 4 : Quantum Entanglement II

Entanglement is a pivotal resource for the development of quantum technologies. It reveals in the Schmidt decomposition of a bipartite state and can be quantified by the purity and/or the entanglement entropies. It can be measured in some simple systems but it remains a *tour de force*.

Density matrix

Hermitian, nonnegative ($\langle \phi|\hat{\rho}_S|\phi\rangle \geq 0$), unit trace

Notion of partial trace : $\hat{\rho}_S = \mathrm{Tr}_E(\hat{\rho}_{SE}) = \sum_m {}_E\langle m|\hat{\rho}_{SE}|m\rangle_E$ with $\{|m\rangle, m\}$ an orthonormal basis of E.

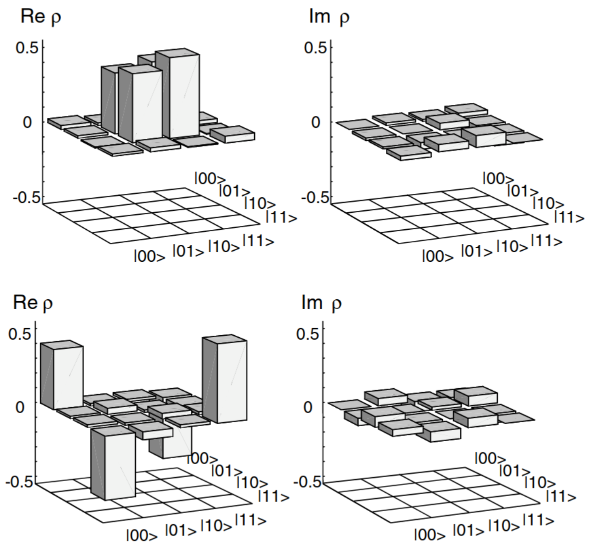
Quantum tomography

The density matrix may be reconstructed using *repeated* measurements

Measurements in the computation basis yields the populations

$$\hat{\rho}_S = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$$

To measure the coherences, measure in a different basis (or rotate the system’s state before measuring in the computation basis)



Lecture 4 : Quantum Entanglement II

Schmidt decomposition

Any state of a bipartite system may be written as

$|\Psi\rangle_{\mathrm{SE}} = \sum_{1\leqslant n\leqslant r} \sqrt{\lambda_n}|u_n\rangle_{\mathrm{S}} \otimes |\chi_n\rangle_{\mathrm{E}}$ where both $\{|u_n\rangle, n\}$ and $\{|\chi_n\rangle, n\}$ form orthonormal sets

The Schmidt decomposition is not unique but the Schmidt rank *r* and spectrum are

The density matrix then reads as $\hat{\rho}_{\mathrm{S}} = \sum_{1\leqslant n\leqslant r} \lambda_n |u_n\rangle\langle u_n|$

How to find the Schmidt decomposition ?

(i) In some (rare) cases, it is obvious…

(ii) Otherwise,

- Compute the reduced density matrix of S (or E) : $\hat{\rho}_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}(|\Psi\rangle\langle\Psi|)$
- Diagonalize it : $\hat{\rho}_{\mathrm{S}} = \rho_u |u\rangle\langle u| + \rho_v |v\rangle\langle v|$ (case study of a qubit, 2-dimensional space)
- Rewrite $|\Psi\rangle_{\mathrm{SE}}$ in the orthonormal basis $\{|u\rangle, |v\rangle\}$: $|\Psi\rangle_{\mathrm{SE}} = \alpha|u\rangle_{\mathrm{S}} \otimes |\chi_u\rangle_{\mathrm{E}} + \beta|v\rangle_{\mathrm{S}} \otimes |\chi_v\rangle_{\mathrm{E}}$

→ The states $|\chi_u\rangle_{\mathrm{E}}$ and $|\chi_v\rangle_{\mathrm{E}}$ are necessarily orthogonal (Schmidt form)

Lecture 4 : Quantum Entanglement II

Entanglement entropies

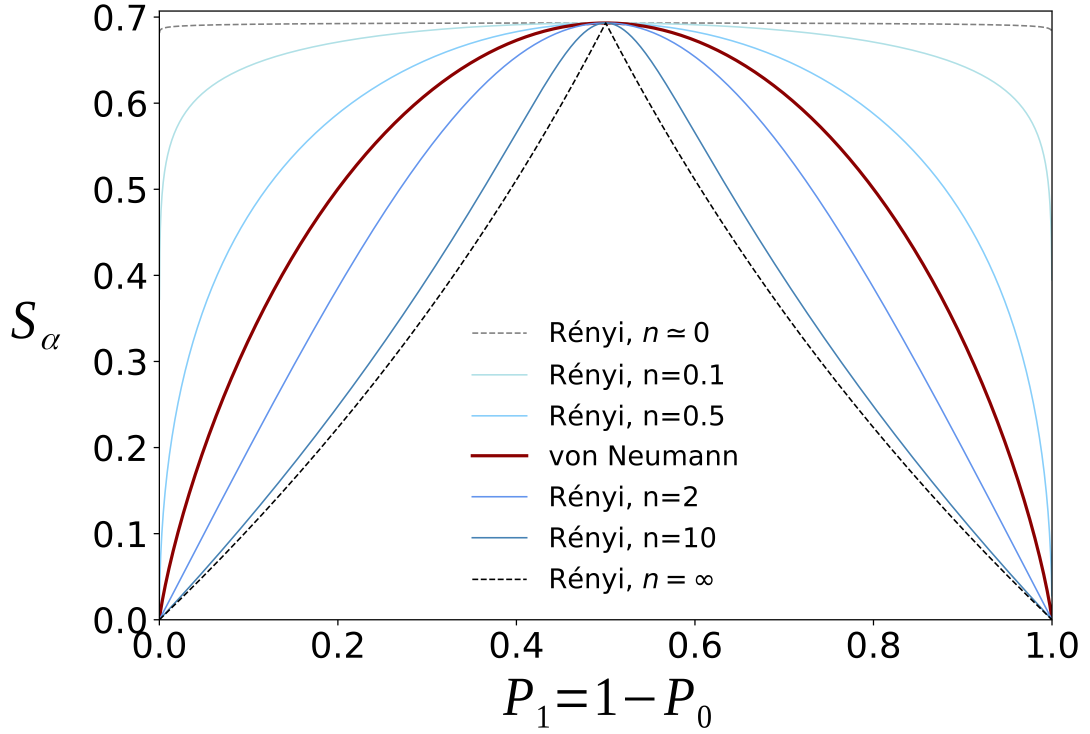
The degree of entanglement may be quantified by the entanglement entropies

von Neumann entropy

$$S_{\mathrm{vN}} = -\mathrm{Tr}\left[\hat{\rho}\log(\hat{\rho})\right]$$

Rényi entropies

$$S_{\alpha} = \frac{1}{1-\alpha}\log\left[\mathrm{Tr}\left(\hat{\rho}^{\alpha}\right)\right]$$



Entanglement entropies (purity, α=2) have been measured in ultracold atoms and trapped ions but it remains a challenge.

Lecture 5 : Quantum Measurement Theory

Projective measurements are insufficient to describe most measurements realized in the context of quantum technologies. The most general measurement processes are described by POVMs.

From PVMs to POVMs

POVMs are found by coupling the system *S* to a meter *M* and then performing a PVM on *M* :

$$|\Psi'\rangle_{S\otimes M}=\hat{U}_{S\otimes M}|\psi\rangle_S\otimes|0\rangle_M=\sum_m\left(\hat{M}_m|\psi\rangle_S\right)\otimes|m\rangle_M$$

where the $|m\rangle$'s form a (orthonormal) measurement basis of *M*.

As regards the state of *S*, a POVM is described by

	Pure state	Mixed state
Probability	$P_m=\|\hat{M}_m\|\psi\rangle\|^2$	$P_m=\mathrm{Tr}\left(\hat{M}_m\hat{\rho}\hat{M}_m^\dagger\right)$
Conditional after-measurement state	$\|\Psi_{\|m}'\rangle=\dfrac{\hat{M}_m\|\psi\rangle}{\|\hat{M}_m\|\psi\rangle\|}$	$\hat{\rho}_{\|m}'=\dfrac{\hat{M}_m\hat{\rho}\hat{M}_m^\dagger}{\mathrm{Tr}\left(\hat{M}_m\hat{\rho}\hat{M}_m^\dagger\right)}$
State after an unread measurement	$\hat{\rho}'=\sum_m\hat{M}_m\|\psi\rangle\langle\psi\|\hat{M}_m^\dagger$	$\hat{\rho}'=\sum_m\hat{M}_m\hat{\rho}\hat{M}_m^\dagger$

The $\hat{M}_m$'s are the Kraus operators (*m* is the result of a POVM and $\hat{M}_m$ describes the evolution of *S* conditional to *m*.

The unitarity of $\hat{U}_{S\otimes M}$ implies the completeness relation

$$\sum_n\hat{M}_m^\dagger\hat{M}_m=\hat{1}$$

Lecture 5 : Quantum Measurement Theory

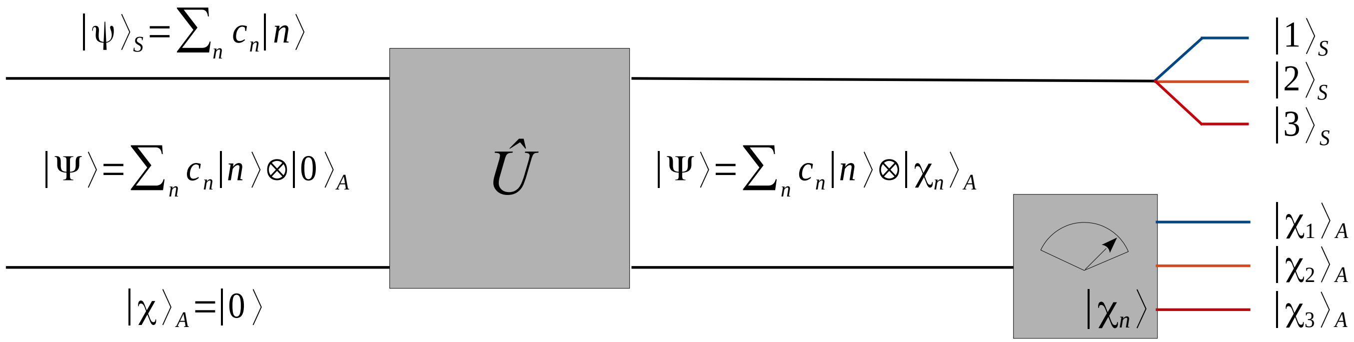
Projective measurement (PVMs) are special cases of generalized measurements (POVMs). They correspond to the case where the Kraus operators $\hat{M}_m$ are projections on mutually orthogonal spaces.

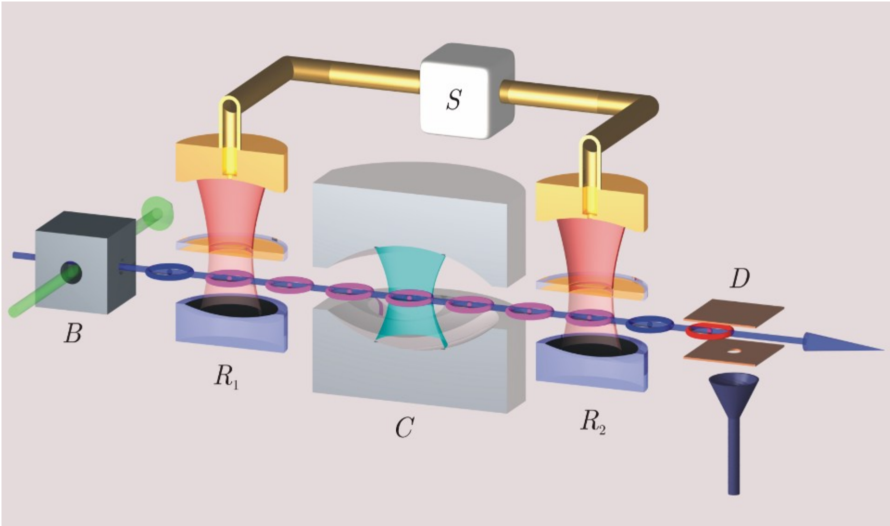
Generalized measurements are pivotal :

Describe realistic measurements (example of imperfect QND measurements, see Homework)

Can outperform PVM measurements (although each Kraus operator yields partial information on the state of S, one may use many more Kraus operators than projectors, see Homework).

Quantum non demolition (QND) measurements



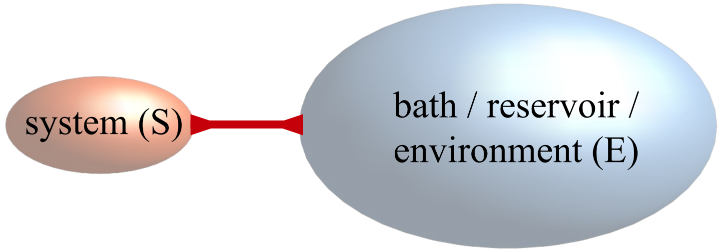


Lecture 6 : Open Quantum Systems I

The dynamics of a system *S* coupled to its environment *E* is described by a quantum map. It takes the universal Kraus form, which may be interpreted as environment-induced jumps of the system's state. In the case where the environment is a bath, characterized by fast relaxation, it yields the differential Lindblad equation upon the Markov approximation.

Kraus operator-sum approach

Evolution of the system's state coupled to its environment

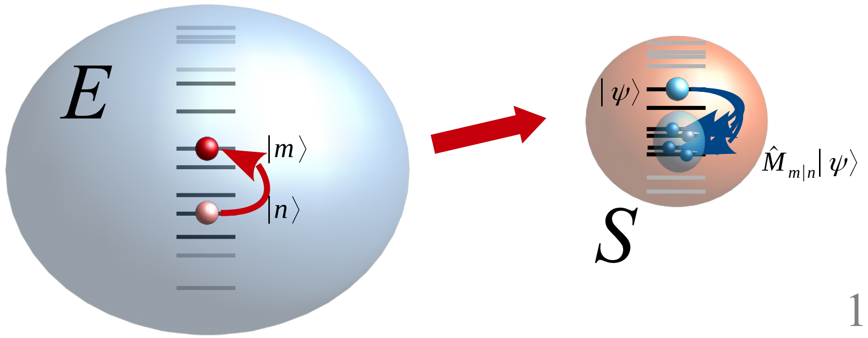


$$\hat{\rho}_{\mathrm{S}}(t)=\sum_{m}\underbrace{{}_{\mathrm{E}}\langle\chi_m|\underbrace{\hat{U}(t)\underbrace{\hat{\rho}_{\mathrm{S}}(0)\otimes\hat{\rho}_{\mathrm{E}}(0)}_{\text{initial state}}\hat{U}(t)^{\dagger}}_{\text{unitary evolution of } S\text{-}E}|\chi_m\rangle_{\mathrm{E}}}_{\text{partial trace over } E}$$

It yields $\hat{\rho}_{\mathrm{S}}(t)=\sum_{m,n}\hat{M}_{m,n}(t)\hat{\rho}_{\mathrm{S}}(0)\hat{M}_{m,n}(t)^{\dagger}$ with $\sum_{m,n}\hat{M}_{m,n}(t)^{\dagger}\hat{M}_{m,n}(t)=1$ and $\hat{M}_{m,n}=\sqrt{\rho_{E,n}(0)}\hat{M}_{m|n}$

The operator sum formula has a straightforward interpretation

$$\hat{\rho}_S(t)=\sum_{m,n}\rho_{E,n}(0)\times P_{m|n}(t)\times\frac{\hat{M}_{m|n}(t)\hat{\rho}_S(0)\hat{M}_{m|n}(t)^{\dagger}}{\mathrm{Tr}_{\mathrm{S}}\left[\hat{M}_{m|n}(t)\hat{\rho}_S(0)\hat{M}_{m|n}(t)^{\dagger}\right]}$$

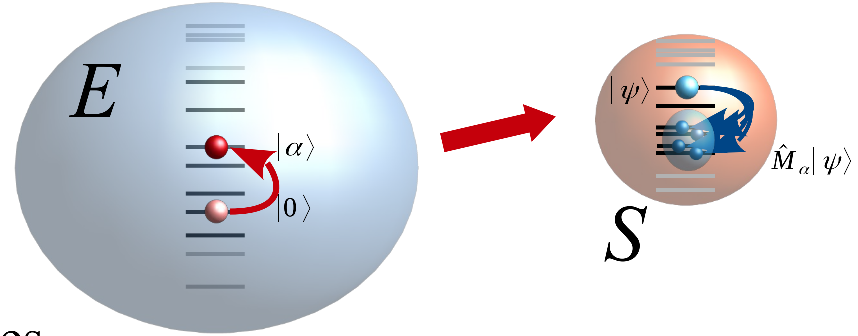


Lecture 6 : Open Quantum Systems I

Kraus theorem

The operator-sum formula is not unique and can always be written as

$$\hat{\rho}_S(t) = \sum_{0 \leqslant \alpha < \dim(\mathcal{E}_S)^2} \hat{M}_\alpha(t)\,\hat{\rho}_S(0)\,\hat{M}_\alpha(t)^\dagger$$



It may be interpreted as the action of a fictitious/effective environment subjected to quantum jumps from a reference state to at most $\dim(\mathcal{E}_S)^2$ states.

Lindblad equation

When the system S is coupled to a *bath* (large system with a huge number of degrees of freedom), the latter relaxes very rapidely to an equilibrium state and looses the memory of previous jumps (Markov process). The dynamics of S is then equivalent to a series of uncorrelated jumps induced by the bath, always in the same reference state. The derivation yields

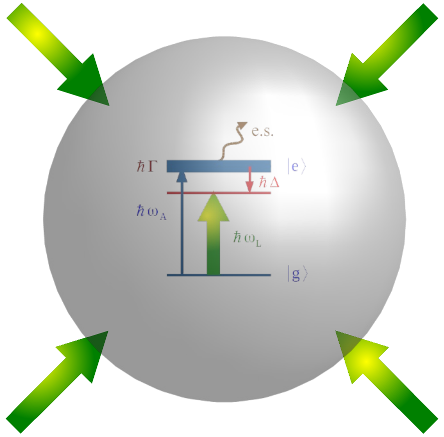
$$\frac{d\hat{\rho}_S}{dt} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}_S] + \sum_{1 \leqslant \alpha < \dim(\text{필}_S)^2} \left\{ \hat{L}_\alpha \hat{\rho}_S \hat{L}_\alpha^\dagger - \frac{1}{2}\hat{\rho}_S \hat{L}_\alpha^\dagger \hat{L}_\alpha - \frac{1}{2}\hat{L}_\alpha^\dagger \hat{L}_\alpha \hat{\rho}_S \right\}$$

There are many applications (damped harmonic oscillator, optical Bloch equations, ….)

Lecture 7 : Open Quantum Systems II

The quantum master equation can be used to describe a variety of open systems, irrespective to the details of the system-bath coupling, and the general form of the Lindblad operators guessed from elementary processes. One may alternatively derive it explicitely from a microscopic approach and get exact expressions of the Lindblad operators.

Optical Bloch equations



Describe the coupling of an atom to a (classical) laser field and the quantized radiation field (assumed in the vacuum)

$$\frac{d\hat{\rho}}{dt}=\frac{1}{i\hbar}\left[\hat{H}_{\mathrm{A}}+\hat{V}_{\mathrm{AL}}(t),\hat{\rho}\right]+\Gamma\left(\hat{\sigma}_{-}\hat{\rho}\hat{\sigma}_{+}-\frac{1}{2}\hat{\rho}\hat{\sigma}_{+}\hat{\sigma}_{-}-\frac{1}{2}\hat{\sigma}_{+}\hat{\sigma}_{-}\hat{\rho}\right)$$

→ Spontaneous emission, damped Rabi oscillations

Derivation of the quantum master equation

(i) Write the exact unitary evolution of the system and the bath

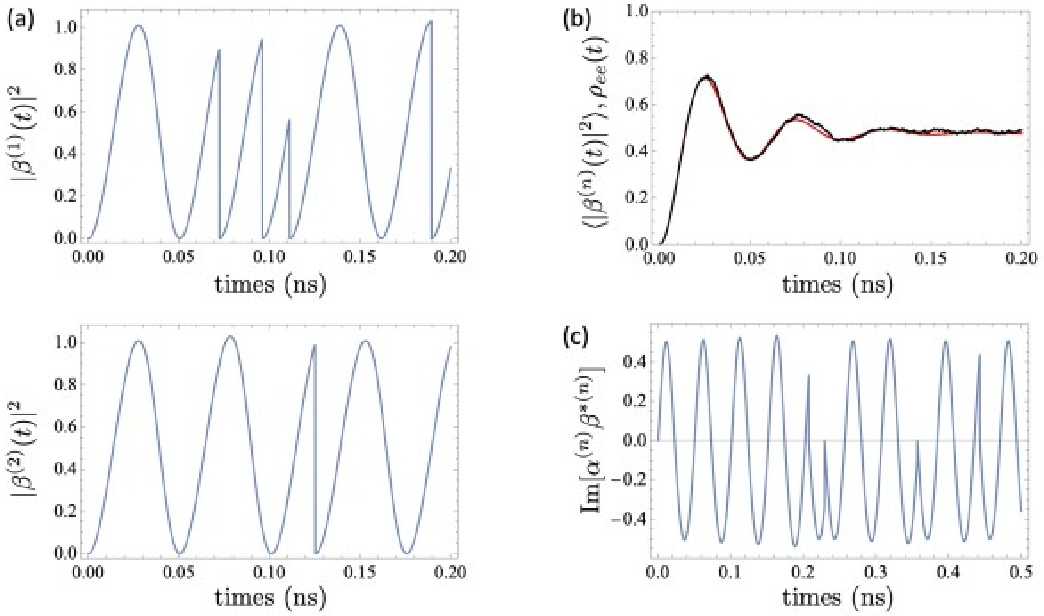
(ii) Restrict to second order in the system-bath coupling (weak-coupling, Born approximation)

(iii) Assume fast relaxation of the bath degrees of freedom and neglect correlations between the system and bath states (Markov approximation)

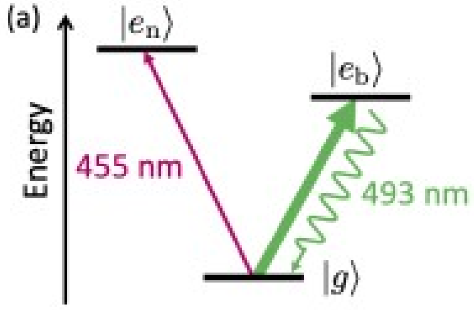
Lecture 8 : Stochastic Wavefunctions

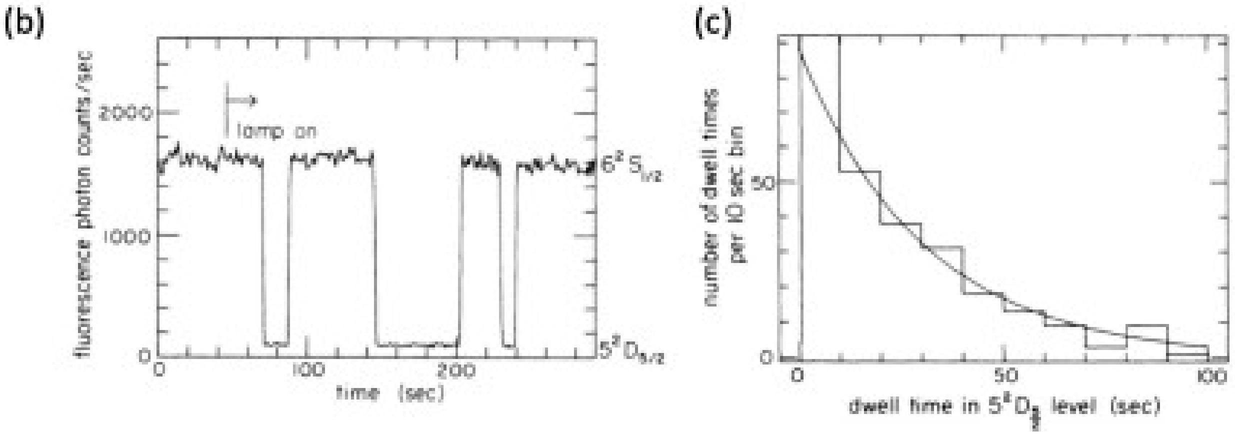
The dynamics of an open quantum system, as described by the quantum master equation, is equivalent to a stochastic-wavefunction approach. The latter describes the dynamics of individual wavefunctions, made of a series of coherent-like (continuous) trajectories interrupted by quantum jumps. Each jump is equivalent to a *read measurement* by the bath. Averaging over all possible outcomes of these measurements, one recovers the quantum master equation (equivalent to a series of *unread measurements*).

Stochastic-wave approach simulating the optical Bloch equations



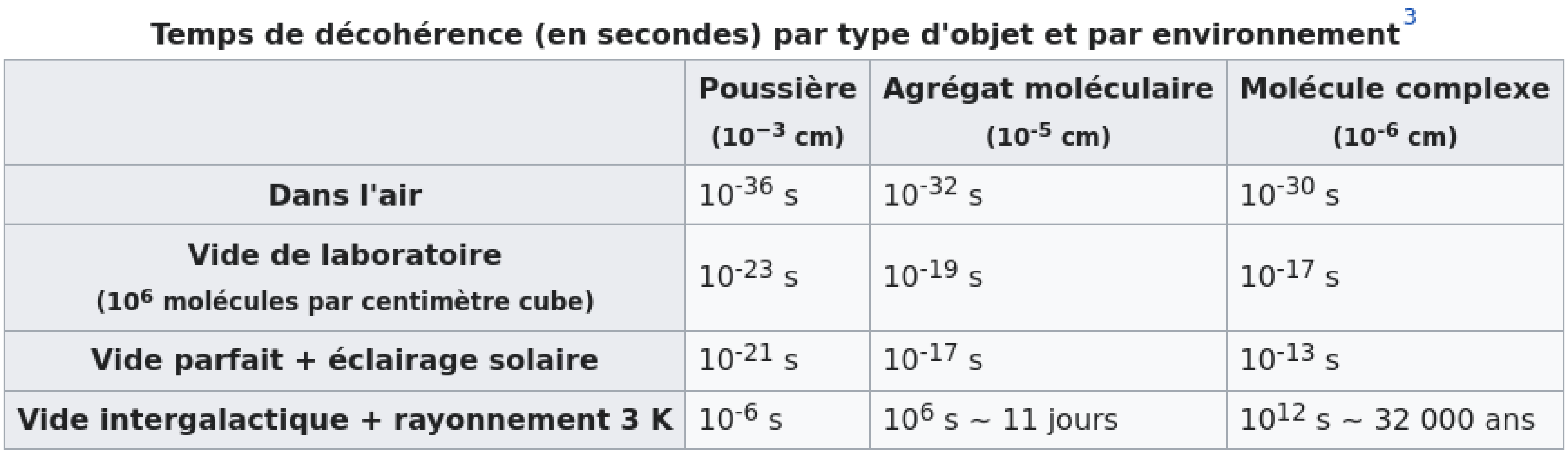
Quantum jumps are a physical reality (first observed in the fluorescence of individual trapped ions)





Lecture 9 : Decoherence

Decoherence is the phenomenon by which quantum superpositions disappear in favor of classical states. It is due to the coupling of the system with its environment, which encompasses a huge number of inaccessible and uncontrolled states, by inducing a strongly entangled state. The state of the system, initially pure, becomes a statistical mixture. The classical states are the pointer states. Pointer states are protected against decoherence. In contrast, superpositions of pointer states are generally not pointer states, which explains the disappearance of their mutual coherence.



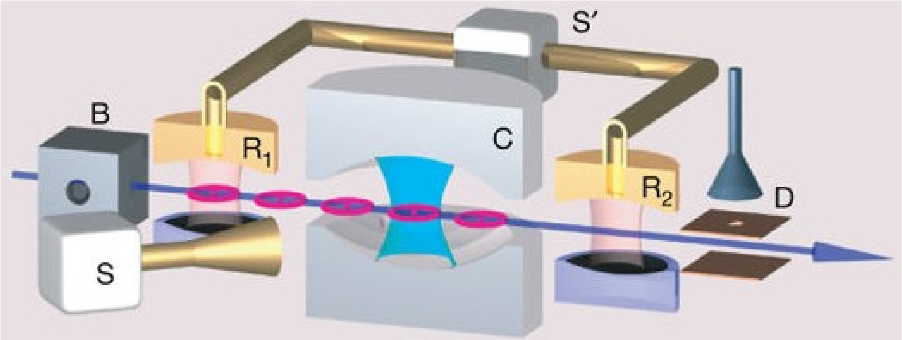
Temps de décohérence (en secondes) par type d'objet et par environnement[3]

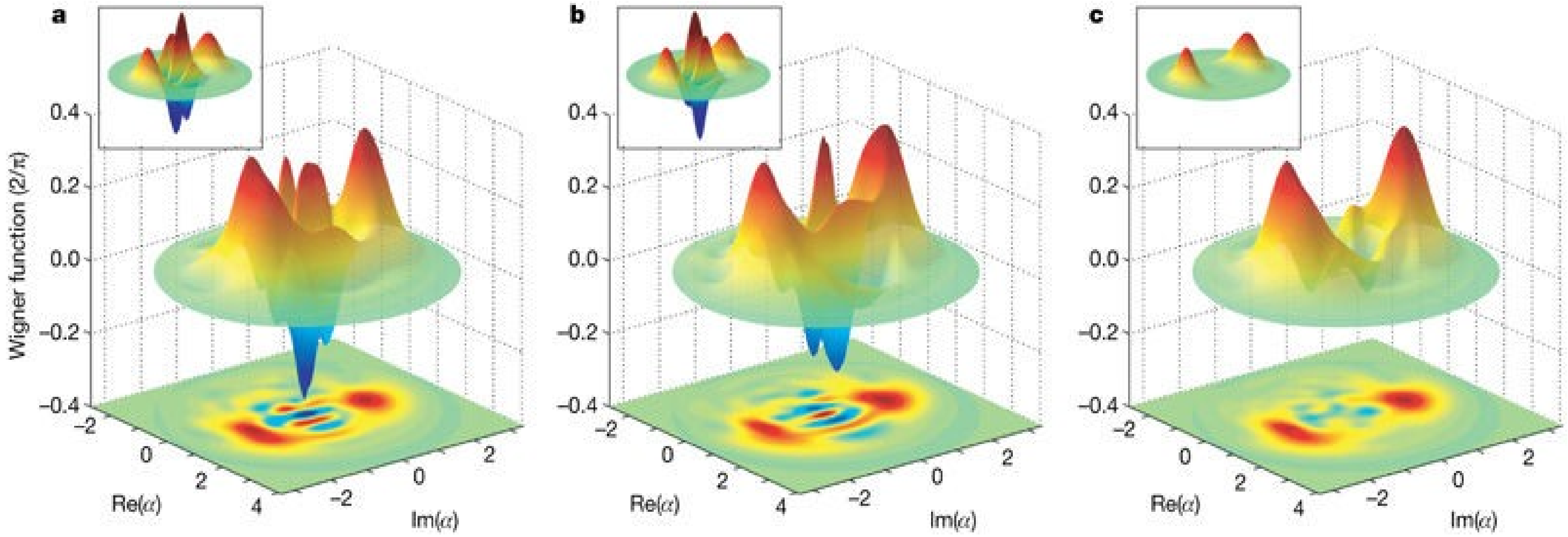
	Poussière (10^{-3} cm)	**Agrégat moléculaire** (10^{-5} cm)	**Molécule complexe** (10^{-6} cm)
Dans l'air	10^{-36} s	10^{-32} s	10^{-30} s
Vide de laboratoire (10^{6} molécules par centimètre cube)	10^{-23} s	10^{-19} s	10^{-17} s
Vide parfait + éclairage solaire	10^{-21} s	10^{-17} s	10^{-13} s
Vide intergalactique + rayonnement 3 K	10^{-6} s	10^{6} s ~ 11 jours	10^{12} s ~ 32 000 ans

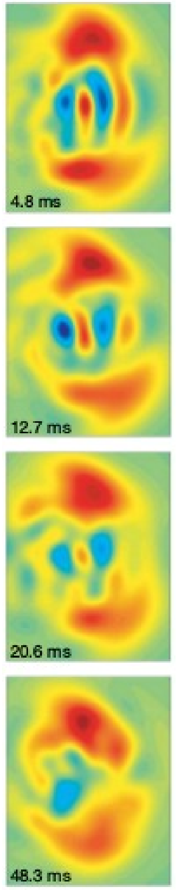
Lecture 9 : Decoherence

Measurement of the density matrix by a kind of tomography technique and representation via the Wigner function

$$W(X,P)=\int dX'\langle X-X'/2|\hat{\rho}|X+X'/2\rangle e^{iX'P}$$







Deléglise *et al.*, *Reconstruction of non-classical cavity field states with snapshots of their decoherence*, Nature **455**, 510-514 (2008)