HW5 Correction

I 1. • $\langle a_\lambda \rangle = \mathrm{Tr}[\rho\, a_\lambda] = \dfrac{1}{Z} \mathrm{Tr}\left[a_\lambda\, e^{-\beta \sum_\mu \hbar\omega_\mu a^\dagger_\mu a_\mu}\right]$, with $Z = \mathrm{Tr}[e^{-\beta H}]$

ρ is diagonal in the $|n_\lambda\rangle$ basis, hence

$$\langle a_\lambda \rangle = \frac{1}{Z} \sum_{n_\lambda} \langle n_\lambda | a_\lambda\, e^{-\beta \sum_\mu \hbar\omega_\mu a^\dagger_\mu a_\mu} | n_\lambda \rangle$$

$$= \frac{1}{Z} \sum_{n_\lambda} \langle n_\lambda | a_\lambda\, e^{-\beta \hbar\omega_\lambda a^\dagger_\lambda a_\lambda} | n_\lambda \rangle$$

$$= \frac{1}{Z} \sum_{n_\lambda} e^{-\beta\hbar\omega_\lambda n_\lambda} \underbrace{\langle n_\lambda | a_\lambda | n_\lambda \rangle}_{=0} = 0.$$

Similarily for $\langle a^\dagger_\lambda \rangle$

• $\langle a^\dagger_\lambda a_\lambda \rangle = \dfrac{1}{Z} \displaystyle\sum_{n_\lambda = 0}^{\infty} e^{-\beta\hbar\omega_\lambda n_\lambda}\, n_\lambda$ and $Z = \displaystyle\sum_{n_\lambda=0}^{\infty} e^{-\beta\hbar\omega_\lambda n_\lambda}$.

$$\Rightarrow \langle a^\dagger_\lambda a_\lambda \rangle = -\frac{d}{d\beta\hbar\omega_\lambda} \ln\left[\sum_{n_\lambda=0}^{\infty} e^{-\beta\hbar\omega_\lambda n_\lambda}\right]$$

$$= -\frac{d}{d\beta\hbar\omega_\lambda} \ln\left[\frac{1}{1-e^{-\beta\hbar\omega_\lambda}}\right] = \frac{1}{e^{\beta\hbar\omega_\lambda}-1}$$

2. – Optical $\hbar\omega_\lambda \sim 1\,\mathrm{eV}$; for n_λ to be non negligible, need $\beta\hbar\omega_\lambda \sim 1$

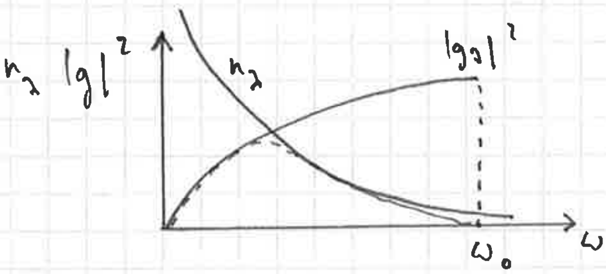
Hence: $T \gtrsim \dfrac{\hbar\omega_\lambda}{k_B} \gtrsim 12000\,\mathrm{K}$!

– μcowave: $\nu = 10\,\mathrm{GHz}$ $\Rightarrow$ $\hbar\omega = h\nu \lesssim k_B T \Rightarrow T \gtrsim 1\,\mathrm{K}$.

3. $\langle R(t)\, R(t-\tau) \rangle = \displaystyle\sum_\lambda \sum_{\lambda'} \left\langle \left(g_\lambda a_\lambda e^{-i\omega t} + g^*_\lambda a^\dagger_\lambda e^{i\omega t}\right)\left(g_{\lambda'} a_{\lambda'} e^{-i\omega(t-\tau)} + g^*_{\lambda'} a^\dagger_{\lambda'} e^{i\omega(t-\tau)}\right)\right\rangle$

$$= \sum_\lambda |g_\lambda|^2 \left[(n_\lambda+1)\, e^{-i\omega\tau} + |g_\lambda|^2 n_\lambda\, e^{i\omega\tau}\right]$$

as $\langle a_\lambda a^\dagger_\lambda \rangle = 1 + n_\lambda$ (remember $[a, a^\dagger] = 1$)



Does not change the idea that $|g_\lambda|^2 n_\lambda$ is a broad function.
Here the typical width would be $\sim k_B T \Rightarrow \tau_c \sim \hbar / k_B T$

II Quantum channel.

2.1 1. $$U_{SR}\,|\psi(0)\rangle = \sqrt{1-p}\;|\psi_S\rangle|\chi_0\rangle + \sqrt{\tfrac{p}{3}}\;\sigma_x|\psi_S\rangle|\chi_x\rangle + \sqrt{\tfrac{p}{3}}\;\sigma_y|\psi_S\rangle|\chi_y\rangle + \sqrt{\tfrac{p}{3}}\;\sigma_z|\psi_S\rangle|\chi_z\rangle$$

2. $M_0 = \sqrt{1-p}\;\hat{\mathbb{1}}$; $M_{x,y,z} = \sqrt{\tfrac{p}{3}}\;\hat{\sigma}_{x,y,z}$

3 $$\rho' = \sum_\alpha M_\alpha \rho M_\alpha^\dagger = (1-p)\,\rho + \frac{p}{3}\sigma_x\rho\sigma_x + \frac{p}{3}\sigma_y\rho\sigma_y + \frac{p}{3}\sigma_z\rho\sigma_z .$$

4 $$\sigma_x\rho\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\rho_{00}&\rho_{01}\\\rho_{10}&\rho_{11}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}\rho_{11}&\rho_{10}\\\rho_{01}&\rho_{00}\end{pmatrix}$$

$$\sigma_y\rho\sigma_y = \begin{pmatrix}\rho_{11}&-\rho_{10}\\-\rho_{01}&\rho_{00}\end{pmatrix} \quad \text{and} \quad \sigma_z\rho\sigma_z = \begin{pmatrix}\rho_{00}&-\rho_{01}\\-\rho_{10}&\rho_{11}\end{pmatrix}$$

Hence : $$\rho' = \begin{bmatrix} 1-\tfrac{2}{3}p\,\rho_{00} + \tfrac{2}{3}p\,\rho_{11} & (1-\tfrac{4}{3}p)\,\rho_{01} \\ (1-\tfrac{4}{3}p)\,\rho_{10} & 1-\tfrac{2}{3}p\,\rho_{11} + \tfrac{2}{3}p\,\rho_{00} \end{bmatrix}$$

2.2 1. $$U_{SR}\,|\psi(0)\rangle = \sqrt{1-p}\;|\psi_S\rangle|\chi_0\rangle + \sqrt{p}\,\hat{\sigma}_z|\psi_S\rangle|\chi_z\rangle$$

2. $M_0 = \sqrt{1-p}\;\hat{\mathbb{1}}$; $M_z = \sqrt{p}\;\hat{\sigma}_z$

3. $$\rho' = M_0\rho M_0^\dagger + M_z\rho M_z^\dagger = (1-p)\,\rho + p\,\sigma_z\rho\sigma_z$$
$$= \begin{bmatrix}\rho_{00} & (1-2p)\rho_{01} \\ (1-2p)\rho_{10} & \rho_{11}\end{bmatrix}$$

4. $p = 1/2 \;\Rightarrow\; \rho'$ = diagonal corresponding to a statistical mixture.
The coherences have disappeared.

5. $$|\psi_{SR}\rangle = \frac{1}{\sqrt{2}}\left(\alpha|0\rangle + \beta|1\rangle\right)|\chi_0\rangle + \frac{1}{\sqrt{2}}\left(\alpha|0\rangle - \beta|1\rangle\right)|\chi_z\rangle$$
$$= \frac{\alpha}{\sqrt{2}}\underbrace{\left(|\chi_0\rangle + |\chi_z\rangle\right)}_{|\phi_0\rangle}|0\rangle + \frac{\beta}{\sqrt{2}}\underbrace{\left(|\chi_0\rangle - |\chi_z\rangle\right)}_{|\phi_1\rangle}|1\rangle$$

$\langle\phi_0|\phi_1\rangle = 0 \;\Rightarrow\;$ S and R are entangled $\Rightarrow$ Tracing over R gives a statistical mixture.

III Dephasing of a qbit

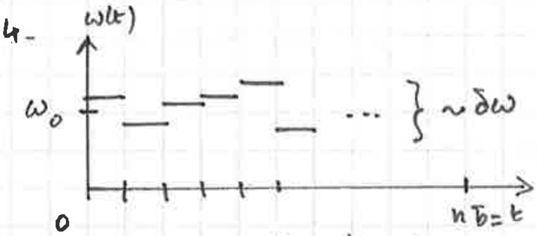
1. Consider 2 hyperfine states of an alkali atom (Rb or Cs), for example: $|0\rangle = |5S_{1/2}; F=1, M=1\rangle$ and $|1\rangle = |5S_{1/2}; F=2\, M=2\rangle$. The transition is sensitive to a magnetic field B: $\Delta\nu \propto B$. A fluctuating $B(t)$ therefore results into $\omega(t) = \omega_0 + \mu B(t)$ due to the Zeeman effect.

2. $H = -\frac{\hbar\omega(t)}{2}\sigma_z \Rightarrow \dot{\alpha} = -i\omega(t)\alpha$ and $\dot{\beta} = i\omega(t)\beta$

$\Rightarrow \alpha(t) = \alpha_0\, e^{-\frac{i}{2}\int_0^t \omega(t')dt'}$ and $\beta = \beta_0\, e^{\frac{i}{2}\int_0^t \omega(t')dt'}$

3. Whatever the temporal profile of the variation of $\omega(t)$ is, $|\alpha|^2 = |\alpha_0|^2$ and $|\beta(t)|^2 = |\beta_0|^2 \Rightarrow$ populations unchanged.

4.



The coherence during τ in ~~same~~ $[j\tau, (j+1)\tau]$ is $\rho_{01}(t) = \alpha\beta^*(t) = \alpha_0\beta_0^*\, e^{-i\omega_0\tau}\, e^{-i\delta\omega_j\tau}$

and step by step $\rho_{01}(t) = \langle \alpha_0\beta_0^*\, e^{-i\omega_0 n\tau} \times e^{-i\delta\omega_1\tau}\, e^{-i\delta\omega_2\tau} \dots \rangle$

if $\delta\omega_1$ is independent of $\delta\omega_2$ ... (Markov...)

$$\rho_{01}(t) = \alpha_0\beta_0^*\, e^{-i\omega_0 t}\, \langle e^{-i\delta\omega\tau}\rangle^n$$

5. $$\langle e^{-i\delta\omega\tau}\rangle = \int e^{-i\delta\omega\tau}\, e^{-\delta\omega^2/2\Delta\omega^2}\frac{d\delta\omega}{\Delta\omega\sqrt{2\pi}}$$

$$= e^{-\frac{\Delta\omega^2\tau^2}{2}}$$

Hence: $$\langle e^{-i\delta\omega\tau}\rangle^n = e^{-\frac{\Delta\omega^2\tau}{2}\times n\tau} = e^{-\frac{\Delta\omega^2\tau}{2}t}.$$

$$T_2 = \frac{2}{\Delta\omega^2\tau} \quad \text{and} \quad \rho_{01}(t) = e^{-i\omega_0 t}\, e^{-t/T_2}\, \alpha_0\beta_0^*$$

6. $\rho_{00} = |\alpha|^2 = \text{cste} \Rightarrow \dot{\rho}_{00} = 0$; $\dot{\rho}_{11} = 0$

$$\dot{\rho}_{01} = -\left(i\omega_0 + \frac{1}{T_2}\right)\rho_{01}$$

7. Dephasing means that you dephase relatively $|0\rangle$ and $|1\rangle$

$\Rightarrow e^{-i\varphi\hat{\sigma}_z}[\alpha|0\rangle + \beta|1\rangle] = \alpha\, e^{-i\varphi}|0\rangle + \beta\, e^{i\varphi}|1\rangle$

The $\hat{\sigma}_z$ is thus precisely the operator that does that!

Hence $\quad L_z = \sqrt{\gamma}\ \sigma_z$, where γ is a rate (unknown at this stage)

8. Bloch in Lindblad form:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}\,[H_0, \rho] + \gamma\left[\sigma_z \rho \sigma_z - \frac{1}{2}\underbrace{\sigma_z^2}_{=\mathbb{1}}\rho - \rho\,\underbrace{\frac{1}{2}\sigma_z^2}_{\mathbb{1}}\right]$$

$$\Rightarrow \quad \frac{d\rho}{dt} = \frac{1}{i\hbar}\,\frac{\hbar}{2}\left[\begin{pmatrix} -\omega_0 & 0 \\ 0 & \omega_0 \end{pmatrix}\rho - \rho\begin{pmatrix} -\omega_0 & 0 \\ 0 & \omega_0 \end{pmatrix}\right]$$

$$+ \gamma\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \rho\right]$$

$$= \begin{pmatrix} 0 & (-i\omega_0 - 2\gamma)\rho_{01} \\ (i\omega_0 - 2\gamma)\rho_{10} & 0 \end{pmatrix}$$

$$\Rightarrow \quad \dot{\rho}_{01} = -(i\omega_0 + 2\gamma)\,\rho_{01} \qquad \Rightarrow \qquad 2\gamma = \frac{1}{T_2}$$

3- Parity measurement and fidelity

1. From lecture 2, we know that the state of a qbit initially in $|0\rangle$ submitted to a coupling $\Omega e^{-i\phi}$ (on resonance) is:

$$|\psi(t)\rangle = \cos\frac{\Omega t}{2}\,|0\rangle + i\sin\frac{\Omega t}{2}\,e^{-i\phi}\,|1\rangle$$

(do not bother about the $-i$ / i $e^{i\phi}$ / $e^{-i\phi}$ ... They are a matter of convention. Use the result given to you!)

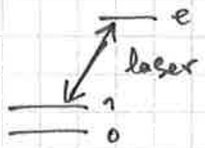
If you apply the wave such that $\frac{\Omega t}{2} = \frac{\pi}{4}$ ($\pi/2 = \Omega t$ pulse)

then $|0\rangle \longrightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}e^{-i\phi}|1\rangle = |0'\rangle$

$|1\rangle \longrightarrow$ a vector $\perp$ to $|0'\rangle$; $\frac{1}{\sqrt{2}}\left(ie^{i\phi}|0\rangle + |1\rangle\right)$ does it!

2. $$\mathcal{F} = \frac{1}{2}\left(\langle 00| + \langle 11|\right)\rho_{exp}\left(|00\rangle + |11\rangle\right)$$
$$= \frac{1}{2}\left(\rho_{00,00} + \rho_{11,11}\right) + \frac{1}{2}\left(\rho_{00,11} + \rho_{11\,00}\right)$$
$$= \frac{1}{2}\left(P_{00} + P_{11}\right) + \rho_{00,11}$$

3. As explained in lecture 2, you can make the ion scatters light. Assume the following level structure, and apply the laser on the 1-e transition.



After the preparation of $|\Psi_+\rangle$, you apply the laser tuned to the 1-e transition. If you measure light scattered, the ion is in $|1\rangle$, if not (dark ion) it is in $|0\rangle$.

You repeat N times and count the number of times you find the two ions in $|0\rangle$ or $|1\rangle$ at the same time. Then

$$P_{00} = \lim_{N\to\infty} \frac{N_{00}}{N} \quad \text{and} \quad \frac{N_{11}}{N} \to P_{11}$$

4. As detailed in lecture 5 when we discussed the tomography of a qbit, applying the same rotation on A and B prior to the measurement amounts to changing the basis states, e.g. $|00\rangle$ ~~into~~ is changed in $R_A(\phi)\otimes R_B(\phi)|00\rangle$, and similarly for the 3 other states.

Hence: $P_{00}(\phi)$ corresponds to the measurt of ρ in the rotated state $R_A\otimes R_B|00\rangle \Rightarrow P_{00}(\phi) = \langle 00| R_A^{-1}\otimes R_B^{-1}\,\rho\, R_A R_B|00\rangle$

You can thus interpret this in 2 ways.

(1) you measure the ~~probability~~ population ~~then~~ of the rotated density matrix $\rho(\phi) = R_A^{-1}R_B^{-1}\,\rho\, R_A R_B$ ~~has a comp~~ in the $|00\rangle$ ~~basis~~ . state.

<u>or</u> (2) you measure the population of the density matrix ρ in the rotated state $R_A R_B|00\rangle$.

5. $$R_A\otimes R_B\left(|01\rangle + |10\rangle\right) = \frac{1}{2}\left(|0\rangle + ie^{-i\phi}|1\rangle\right)\left(ie^{i\phi}|0\rangle + |1\rangle\right) + \frac{1}{2}\left(ie^{i\phi}|0\rangle + |1\rangle\right)\left(|0\rangle + ie^{-i\phi}|1\rangle\right)$$

$$= ie^{i\phi}|00\rangle + ie^{-i\phi}|11\rangle$$

Hence: $$(P_{01}+P_{10})(\phi) = \left(\langle 01| + \langle 10|\right) R_A^{-1} R_B^{-1}\,\rho\, R_A R_B\left(|01\rangle + |10\rangle\right)$$

$$= P_{00} + P_{11} + \rho_{0011} + \rho_{1100}$$

Same (tedious ...) calculation for $P_{00}(\phi) + P_{11}(\phi)$

6. from the figure, we get the amplitude $2\rho_{00,11}$ of the oscillation: $4\rho_{00,11} \approx 1{,}56 \Rightarrow \rho_{00,11} \approx 0{,}39$

7. $\mathcal{F} = \frac{1}{2}(0{,}43 + 0{,}46) + 0{,}39 \approx 0{,}83$

8. $\mathcal{F} = |\langle \psi_t | \psi \rangle|^2$ where $|\psi_t\rangle = |\psi_+\rangle$ (The target state) and $|\psi\rangle$ the state produced in the experiment.

This is simply writing that for a pure state in the experiment

$$\rho_{exp} = |\psi\rangle\langle\psi| \Rightarrow \langle \psi_+ | \rho | \psi_+ \rangle = |\langle \psi_+ | \psi \rangle|^2$$

Hence: $\mathcal{F} = \frac{1}{2}\left| \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \right|^2 = \frac{1}{2}|ac + bd|^2$

9. $$|a|^2 + |c|^2 = (|a| - |c|)^2 + 2|ac|$$
$$|b|^2 + |d|^2 = (|b| - |d|)^2 + 2|bd|$$
$$\Rightarrow 2 \geq 2|ac| + |bd| \Rightarrow |ac| + |bd| \leq 1$$

10. $$(|ac| + |bd|)^2 = |ac|^2 + |bd|^2 + 2|abcd| \leq 1$$

But $\mathcal{F} = \frac{1}{2}(|ac|^2 + |bd|^2 + 2|abcd|\cos\varphi)$

$\hookrightarrow$ argument de $a^*b c^* d$

$$2\mathcal{F} \leq |ac|^2 + |bd|^2 + 2|abcd| \leq 1$$

(as $\cos\varphi \leq 1$)

Hence nonentangled $\Rightarrow \mathcal{F} \leq 1/2$

Therefore. $\mathcal{F} > 1/2 \Rightarrow$ entangled.

11. $\mathcal{F} = \langle \psi_+ | \rho | \psi_+ \rangle = \sum_k p_k \langle \psi_+ | \rho_k | \psi_+ \rangle$.

if $\rho_k = |\psi_k\rangle\langle\psi_k|$ with $|\psi_k\rangle$ separable, the previous questions show that $\langle \psi_+ | \rho_k | \psi_+ \rangle \leq 1/2 \Rightarrow \sum_k p_k \langle \psi_+ | \rho_k | \psi_+ \rangle \leq \frac{1}{2} \sum p_k$

$= 1/2$