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Lecture 6: Open quantum systems I: Kraus-Lindblad approach

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This chapter initiates our discussion of open quantum systems, which will also be the subject of the next chapters. Here the term "open system" refers to any system that is not strictly speaking isolated, and, contrary to standard terminology in thermodynamics, no distinction is made here between closed and open systems[1]. In practice, it is impossible to completely isolate a system from its environment, whatever it is. When the system is large enough and the coupling is weak, this generally leads to an equilibrium state where the only effect of the environment is to impose its temperature and chemical potential to the system. In contrast, for individual quantum systems, which may be microscopic or mesoscopic, the dynamical time scales may be long enough to be observed and it is relevant to consider the time-dependent dynamics of the system coupled to its environment before an equilibrium state is reached. This is fortunate because otherwise we would not be able to exploit the very quantum properties of the system, in particular quantum superposition states and entanglement.

In fact, the environment, whatever it is, strongly affects the behaviour of almost any individual quantum system. Just think about the simplest quantum system you can imagine, say an atom in the vacuum. Its coupling to the radiation field makes the atomic excited states unstable and generates spontaneous emission. More generally, coupling a system of interest to another system usually strongly impacts its dynamics, which is no longer unitary. One of the most spectacular effects of coupling a system to its environment is *decoherence*, which suppresses quantum coherence and entanglement. This phenomenon will be more specifically discussed in lecture 8.

The study of open systems is thus of utmost importance to modern quantum science and technologies. There are several quite different, but in fine equivalent, approaches to open quantum systems. In this chapter, we discuss a formal approach using the so-called *Kraus formalism*. It allows us to derive the most general and compact form of the evolution of an open quantum system, coupled to any other system. It leads to the so-called concept of *completely positive maps*. We then discuss the case where the system of interest is coupled more specifically to a bath in the thermodynamic sense. In this case, the bath relaxes rapidly towards its equilibrium state and the dynamics of the system becomes *Markovian*, i.e. its instantaneous evolution only depends on its state at the considered instant, the system losing the memory of its previous evolution. We hence

[1]We recall that in thermodynamics a system is said to be closed when it only exchanges energy with its environment and open when it additionally exchanges matter.

obtain the *Lindblad equation*, which is the most general form of the Markovian evolution of the density matrix of an open quantum system. Finally, we discuss two usecases: the damped harmonic oscillator and a driven open system.

1 Dynamics of a quantum system coupled to another quantum system: Kraus formalism

Here we consider a quantum system $\mathcal{S}$ coupled to another system $\mathcal{B}$, on which we do not make any particular assumptions at the moment. We, however, assume that (i) the two systems are prepared independently, so that the initial density matrix of the bipartite system $\mathcal{S} \otimes \mathcal{B}$ factorizes at time $t = 0$,

$$\hat{\rho}_{\mathcal{S}\otimes\mathcal{B}}(0) = \hat{\rho}_{\mathcal{S}}(0) \otimes \hat{\rho}_{\mathcal{B}}(0) \ , \tag{1}$$

and (ii) $\mathcal{S} \otimes \mathcal{B}$ is isolated during the subsequent evolution. In principle, the coupling of the two systems generally leads to an entangled state so that the dynamics of one cannot be separated from that of the other. If we are only interested in the dynamics of the system $\mathcal{S}$, i.e. if we only plan to measure $\mathcal{S}$ independently of $\mathcal{B}$, we can nevertheless get rid of the latter by performing a partial trace on its degrees of freedom. We shall see that this operation strongly alters the dynamics of the system $\mathcal{S}$.

1.1 Preliminary: Evolution of an isolated system

Consider first an isolated system. Its state is described by a ket $|\psi(t)\rangle$. The solution of the Schrödinger equation reads as

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle \ , \tag{2}$$

where $|\psi(0)\rangle$ is the initial state and $\hat{U}(t)$ is the evolution operator, solution of the equation

$$i\hbar\frac{d\hat{U}}{dt} = \hat{H}(t)\hat{U}(t) \qquad \text{with} \qquad \hat{U}(0) = \hat{\mathbb{1}} \ . \tag{3}$$

Here $\hat{H}(t)$ is the system Hamiltonian, which may possibly depend on time.

The density matrix at time t reads as

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \hat{U}(t)\,|\psi(0)\rangle\langle\psi(0)|\,\hat{U}(t)^\dagger \ , \tag{4}$$

that is

$$\boxed{\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}(t)^\dagger} \ , \tag{5}$$

where $\hat{\rho}(0) = |\psi(0)\rangle\langle\psi(0)|$ is the initial density matrix. Moreover, the dynamical equation for the density matrix is found writing

$$i\hbar\frac{d\hat{\rho}}{dt} = \hat{H}\hat{U}\hat{\rho}(0)\hat{U}^\dagger - \hat{U}\hat{\rho}(0)\hat{U}^\dagger\hat{H} \ , \tag{6}$$

that is

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}\big[\hat{H}(t), \hat{\rho}(t)\big]} \ . \tag{7}$$

This formula is nothing but the counterpart of the standard Schrödinger equation in the density matrix formalism.

Equations (5) and (7) still hold when the system $\mathcal{S}$ is in a mixed state but does not interact with its environment. To show this, consider two systems $\mathcal{S}$ and $\mathcal{B}$, initially in an entangled state, which we write in Schmidt form,

$$|\Psi(0)\rangle_{\mathcal{S}\otimes\mathcal{B}} = \sum_n \lambda_n |\psi_n\rangle_{\mathcal{S}} \otimes |\chi_n\rangle_{\mathcal{B}} \ . \tag{8}$$

They interacted in the past to create this entangled state but we assume that they do not interact any longer at $t \geq 0$. The Hamiltonian of the bipartite system then reads as $\hat{H} = \hat{H}_{\mathcal{S}} + \hat{H}_{\mathcal{B}}$. These operators are possibly time dependent although the time dependence is not explicitly written. The operators $\hat{H}_{\mathcal{S}}$ and $\hat{H}_{\mathcal{B}}$ are the Hamiltonians of the systems $\mathcal{S}$ and $\mathcal{B}$ respectively. Since they act in distinct Hilbert spaces, they commute with each other so that the evolution operator reads as $\hat{U}_{\mathcal{S}\otimes\mathcal{B}} = \hat{U}_{\mathcal{S}} \otimes \hat{U}_{\mathcal{B}}$. The bipartite state at time t thus reads as

$$|\Psi(t)\rangle_{\mathcal{S}\otimes\mathcal{B}} = \sum_n \lambda_n \left(\hat{U}_{\mathcal{S}} |\psi_n\rangle\right)_{\mathcal{S}} \otimes \left(\hat{U}_{\mathcal{B}} |\chi_n\rangle\right)_{\mathcal{B}} \ . \tag{9}$$

The family of states $\{|\chi_n\rangle, n\}$ being orthonormal and the operator $\hat{U}_{\mathcal{B}}$ being unitary, the family $\{(\hat{U}_{\mathcal{B}} |\chi_n\rangle), n\}$ is also orthonormal, and the reduced density matrix of the system $\mathcal{S}$ at time t reads as

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_n |\lambda_n|^2 \left(\hat{U}_{\mathcal{S}} |\psi_n\rangle\right)\left(\langle\psi_n| \hat{U}_{\mathcal{S}}^\dagger\right) = \hat{U}_{\mathcal{S}}\hat{\rho}_{\mathcal{S}}(0)\hat{U}_{\mathcal{S}}^\dagger \ . \tag{10}$$

This is nothing but Eq. (5) for the system $\mathcal{S}$. Equation (7) for system $\mathcal{S}$ directly follows.

1.2 Evolution of an open system: The case of a POVM

The first example of coupling a system to another system is that of a generalized measurement, which we named POVM (for more details, see lecture 5). Here the system $\mathcal{B}$ is the meter M, which is assumed to be initially in some reference state $|0\rangle_M$. Then, for any pure state of the system $\mathcal{S}$, the initial state of the bipartite $\mathcal{S}$-M system is

$$|\Psi\rangle_{S\otimes M} = |\psi\rangle_S \otimes |0\rangle_M = |0\rangle_M \otimes |\psi\rangle_S \ . \tag{11}$$

When the system and the meter interact, we get a generally entangled-state, which may be written as

$$|\Psi'\rangle_{S\otimes M} = \hat{U}\,|\Psi\rangle_{S\otimes M} = \sum_m \left(\hat{M}_m\,|\psi\rangle_S\right) \otimes |m\rangle_M \ , \tag{12}$$

where the states $|m\rangle_M$ describe an orthonormal basis of the meter's Hilbert space and the Kraus operators are

$$\hat{M}_m = {}_M\langle m|\,\hat{U}\,|0\rangle_M \ . \tag{13}$$

Here, we wrote

$$\hat{U}\,|\Psi\rangle_{S\otimes M} \ = \ \hat{U}\,|\psi\rangle_S \otimes |0\rangle_M \ = \ \hat{U}\,|0\rangle_M\,|\psi\rangle_S \ = \ \sum_m |m\rangle_M \underbrace{\langle m|_M\,\hat{U}\,|0\rangle_M}_{\hat{M}_m}\,|\psi\rangle_S \ .$$

Note that $\hat{M}_m$ is an operator on $\mathcal{E}_S$.

The Kraus operators $\hat{M}_m$ fulfill a completeness relation, inherited from the unitarity of the evolution operator $\hat{U}$. Indeed, we have

$$1 = \langle\Psi'|\,|\Psi'\rangle = \sum_m {}_S\langle\psi|\,\hat{M}_m^\dagger \hat{M}_m\,|\psi\rangle_S \ , \tag{14}$$

since the states $|m\rangle_M$ form an orthonormal family. This relation being valid for any state of the system S, it implies

$$\boxed{\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{\mathbb{1}}_S} \ . \tag{15}$$

Note that in general the Kraus operators $\hat{M}_m$ are not Hermitian.

Generalized measurement of a pure state

When we read the result of the measurement, we perform a PVM on the meter in the basis $\{|m\rangle_M\}$. It is described by the projector $\hat{\mathcal{P}}_m = \hat{\mathbb{1}}_S \otimes |m\rangle\langle m|_M$. The after measurement state of the bipartite system $\mathcal{S}$-M is then

$$|\Psi'_{|m}\rangle = \frac{\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M}{\left|\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M\right|} \ . \tag{16}$$

This state being a product state, the system $\mathcal{S}$ is well-defined by the ket

$$\boxed{|\psi'_{|m}\rangle = \frac{\hat{M}_m|\psi\rangle}{\left|\hat{M}_m|\psi\rangle\right|}} \ , \tag{17}$$

where now omit the system's index.

According to the Born rule, the probability of measuring m is $P_m = \left|\hat{M}_m|\psi\rangle \otimes |m\rangle_M\right|^2$, that is

$$\boxed{P_m = \left|\hat{M}_m|\psi\rangle\right|^2} . \tag{18}$$

Now, if the result of the measurement is unread, the after-measurement state of S is described by the density matrix

$$\hat{\rho}\,' = \sum_m P_m \left|\psi'_{|m}\right\rangle\!\left\langle\psi'_{|m}\right| = \sum_m P_m \frac{\hat{M}_m |\psi\rangle\langle\psi| \hat{M}_m^\dagger}{\left|\hat{M}_m|\psi\rangle\right|^2} , \tag{19}$$

that is

$$\boxed{\hat{\rho}\,' = \sum_m \hat{M}_m |\psi\rangle\langle\psi| \hat{M}_m^\dagger} . \tag{20}$$

These formulas directly generalize those relevant to PVMs by replacing the projection operators $\hat{\mathcal{P}}_j$ by the Kraus operators $\hat{M}_m$ or $\hat{M}_m^\dagger$ depending on whether they are associated to a ket or a bra.

Generalized measurement of a mixed state

These results can be further generalized to the case where the system $\mathcal{S}$ is in a mixed state, described by the density matrix $\hat{\rho}$. This can be shown using a purification approach and applying the results above (see lecture 5).

The after-measurement state of $\mathcal{S}$ conditionned to the result of the measurement m is

$$\boxed{\hat{\rho}'_{|m} = \frac{\hat{M}_m \hat{\rho} \hat{M}_m^\dagger}{\mathrm{Tr}\big(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger\big)}} , \tag{21}$$

and the probability of measuring m is

$$\boxed{P_m = \mathrm{Tr}\big(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger\big) = \mathrm{Tr}\big(\hat{E}_m \hat{\rho}\big)} , \tag{22}$$

where $\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$.

If the measurement is unread, we find $\hat{\rho}^{'} = \sum_m P_m \times \hat{\rho}'_{|m}$. Combining Eqs. (21) and (22), it yields

$$\boxed{\hat{\rho}' = \sum_m \hat{M}_m \hat{\rho} \hat{M}_m^\dagger} . \tag{23}$$

1.3 Evolution of an open system

To find the evolution of the reduced density matrix of the system $\mathcal{S}$, we first write the evolution of the bipartite system $\mathcal{S} \otimes \mathcal{B}$ and then trace over the degrees of freedom of $\mathcal{B}$.

It yields

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_m {}_{\mathcal{B}}\langle\chi_m|\ \hat{U}_{\mathcal{S}\otimes\mathcal{B}}\ \hat{\rho}_{\mathcal{S}}(0)\otimes\hat{\rho}_{\mathcal{B}}(0)\ \hat{U}^\dagger_{\mathcal{S}\otimes\mathcal{B}}\ |\chi_m\rangle_{\mathcal{B}}\ , \tag{24}$$

where $\hat{U}_{\mathcal{S}\otimes\mathcal{B}}$ is the evolution operator of the bipartite system between the times 0 and t, and $\{|\chi_m\rangle\}$ is any orthonormal basis of the Hilbert space of $\mathcal{B}$. We then choose the latter as an eigenbasis of the initial reduced density matrix of $\mathcal{B}$, so that

$$\hat{\rho}_{\mathcal{B}}(0) = \sum_n \rho_{\mathcal{B},n}(0)\,|\chi_n\rangle\langle\chi_n|_{\mathcal{B}}\ . \tag{25}$$

Inserting Eq. (25) into Eq. (24), we find

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_{m,n}\ {}_{\mathcal{B}}\langle\chi_m|\hat{U}_{\mathcal{S}\otimes\mathcal{B}}|\chi_n\rangle_{\mathcal{B}}\ \hat{\rho}_{\mathcal{S}}(0)\ \rho_{\mathcal{B},n}(0)\ {}_{\mathcal{B}}\langle\chi_n|\hat{U}^\dagger_{\mathcal{S}\otimes\mathcal{B}}|\chi_m\rangle_{\mathcal{B}}\ , \tag{26}$$

that is

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_{m,n}\rho_{\mathcal{B},n}(0)\hat{M}_{m|n}\hat{\rho}_{\mathcal{S}}(0)\hat{M}^\dagger_{m|n} \qquad \text{with} \quad \hat{M}_{m|n} = {}_{\mathcal{B}}\langle\chi_m|\hat{U}_{\mathcal{S}\otimes\mathcal{B}}|\chi_n\rangle_{\mathcal{B}}\ , \tag{27}$$

or, equivalently,

$$\boxed{\hat{\rho}_{\mathcal{S}}(t) = \sum_{m,n}\hat{M}_{m,n}\hat{\rho}_{\mathcal{S}}(0)\hat{M}^\dagger_{m,n} \qquad \text{with} \quad \hat{M}_{m,n} = \sqrt{\rho_{\mathcal{B},n}(0)}\,\underbrace{{}_{\mathcal{B}}\langle\chi_m|\hat{U}_{\mathcal{S}\otimes\mathcal{B}}|\chi_n\rangle_{\mathcal{B}}}_{\equiv\hat{M}_{m|n}}}\ . \tag{28}$$

This formula is called a *positive map* for it transforms a semi-definite positive operator into another one, $\hat{\rho}_{\mathcal{S}}(0) \longrightarrow \hat{\rho}_{\mathcal{S}}(t)$. It may also be called a *quantum map*, a *quantum operation* or a *quantum channel*. Above, we introduced the operator $\hat{M}_{m|n}$ which we shall use later. It is nothing but the operator $\hat{M}_{m,n}$ conditioned to the fact that the system $\mathcal{B}$ was in the state $|\chi_n\rangle$ at time $t = 0$ with unit probability, i.e. $\rho_{\mathcal{B},n}(0) = 1$ and $\rho_{\mathcal{B},n'\neq n}(0) = 0$.

In the case where the system $\mathcal{B}$ is prepared in a pure state, say $\hat{\rho}_{\mathcal{B}}(0) = |0\rangle\langle 0|_{\mathcal{B}}$, the evolution equation (28) can be simplified using $\rho_{\mathcal{B},n} = \delta_{n,0}$, which yieds

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_m \hat{M}_m\hat{\rho}_{\mathcal{S}}(0)\hat{M}^\dagger_m\ , \tag{29}$$

with $\hat{M}_m \equiv \hat{M}_{m,0} = {}_{\mathcal{B}}\langle\chi_m|\hat{U}_{\mathcal{S}\otimes\mathcal{B}}|0\rangle_{\mathcal{B}}$.

Content of the Kraus formula

Equation (29) is nothing but Eq. (20) when we discussed the coupling of a system $\mathcal{S}$ with a measuring device in POVMs. This makes sense since it was the same problem where the measuring device, assumed to be in a well-defined reference state $|0\rangle_{\mathcal{B}}$, is here

replaced by any system $\mathcal{B}$ that we did not specify. In fact Eq. (28) generalizes Eq. (29) to the case where the system $\mathcal{B}$ is in any state, possibly a mixed state.

The operators $\hat{M}_{m,n}$ are called *Kraus operators* just as in POVMs. Note that they are operators acting in the Hilbert space $\mathcal{E}_\mathcal{S}$ of the system $\mathcal{S}$. Moreover, they fulfill the completeness relation

$$\boxed{\sum_{m,n} \hat{M}_{m,n}^\dagger \hat{M}_{m,n} = \hat{\mathbb{1}}_\mathcal{S}} \,. \tag{30}$$

This is a direct consequence of (i) the unitary character of the evolution operator $\hat{U}_{\mathcal{S}\otimes\mathcal{B}}$ and (ii) the normalization of the density matrix of the system $\mathcal{B}$.

We have thus obtained that the evolution of an open quantum system $\mathcal{S}$ is governed by the set of Kraus operators $\hat{M}_{m,n}$. The latter depend on the coupling (all contained in the conditional operators $\hat{M}_{m|n}$) and on the initial state of the bath [contained in the terms $\rho_{\mathcal{B},n}(0)$], but not on the initial state of the system $\mathcal{S}$. The evolution equation (28) of the density operator of an open system also generalizes Eq. (5), valid for an isolated system. In the case of an open system, the unique evolution operator $\hat{U}$ is replaced by the set of Kraus operators. The latter are in general nonunitary. In fact, the completeness relation (30) shows that a Kraus operator is unitary if and only if it is unique. Note that in general, the individual terms $\hat{M}_{m,n}^\dagger \hat{M}_{m,n}$ in Eq. (30) are not simply proportional to the identity either, see examples in the context of POVMs as well as in Sec. 3 below.

Physical interpretation of the Kraus formula

The general map derived above has a straightforward and enlightening interpretation. To see this, let us first compute the probability that the system $\mathcal{B}$ jumps from the state $|\chi_n\rangle$ to the state $|\chi_m\rangle$ after interaction with $\mathcal{S}$ for a time t. We find it by writing the probability that $\mathcal{B}$ is in the state $|\chi_m\rangle$ conditional to it being initially in the state $|\chi_n\rangle$, $P_{m|n}$ with $\hat{\rho}_{\mathcal{S}\otimes\mathcal{B}}(0) = \hat{\rho}_\mathcal{S}(0) \otimes |\chi_n\rangle\langle\chi_n|_\mathcal{B}$. It yields

$$\begin{aligned} P_{m|n} &= \mathrm{Tr}_{\mathcal{S}\mathcal{B}} \left[\hat{\rho}_{\mathcal{S}\otimes\mathcal{B}}(t) \left(|\chi_m\rangle\langle\chi_m|\right)_\mathcal{B}\right] \\ &= \mathrm{Tr}_\mathcal{S} \left[\mathrm{Tr}_\mathcal{B} \left(\hat{U}_{\mathcal{S}\otimes\mathcal{B}} \hat{\rho}_\mathcal{S}(0) \otimes |\chi_n\rangle_\mathcal{B} \langle\chi_n| \hat{U}_{\mathcal{S}\otimes\mathcal{B}}^\dagger |\chi_m\rangle_\mathcal{B} \langle\chi_m|\right)\right] \\ &= \mathrm{Tr}_\mathcal{S} \left({}_\mathcal{B}\langle\chi_m| \hat{U}_{\mathcal{S}\otimes\mathcal{B}} \hat{\rho}_\mathcal{S}(0) \otimes |\chi_n\rangle_\mathcal{B} \langle\chi_n| \hat{U}_{\mathcal{S}\otimes\mathcal{B}}^\dagger |\chi_m\rangle_\mathcal{B}\right), \end{aligned}$$

that is

$$P_{m|n} = \mathrm{Tr}_\mathcal{S} \left[\hat{M}_{m|n} \hat{\rho}_\mathcal{S}(0) \hat{M}_{m|n}^\dagger\right] \,, \tag{31}$$

where $\hat{M}_{m|n}$ is the conditional Kraus operator introduced in Eq. (28). Hence, by virtue of the cyclic property of the trace, we find that the operator $\hat{E}_{m|n} = \hat{M}_{m|n}^\dagger \hat{M}_{m|n}$ measures the probability that the system $\mathcal{B}$ jumps from the state $|\chi_n\rangle$ to the state $|\chi_m\rangle$. Quite counterintuitively, this operator acts in the Hilbert space of $\mathcal{S}$ and not of $\mathcal{B}$. This, however,

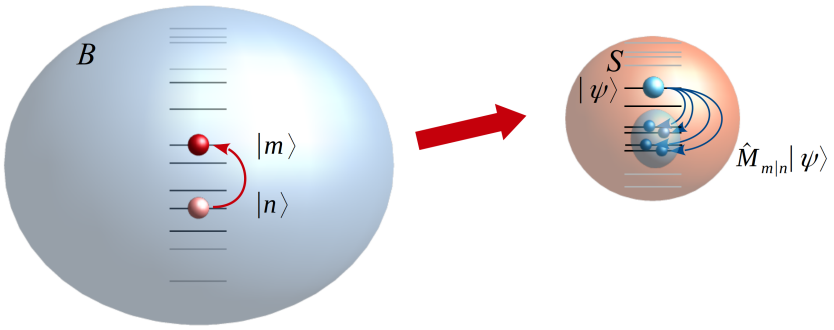


Figure 1: Schematic representation of the action of a conditional Kraus operator $\hat{M}_{m|n}$. A jump from $|n\rangle$ to $|m\rangle$ in system $\mathcal{B}$ induces a jump from $|\psi\rangle$ to $\hat{M}_{m|n}|\psi\rangle$ (nonnormalized) in the system $\mathcal{S}$.

makes sense: Since a jump of the state of $\mathcal{B}$ is induced by its coupling with $\mathcal{S}$, it is natural that its probability depends on the state of the latter.

We may now rewrite the quantum map (27) by introducing the ratio of the left and right hand side terms in Eq. (31), which yields

$$\hat{\rho}_{\mathcal{S}}(t) = \sum_{m,n} \rho_{\mathcal{B},n}(0) \times P_{m|n} \times \frac{\hat{M}_{m|n}\hat{\rho}_{\mathcal{S}}(0)\hat{M}_{m|n}^{\dagger}}{\mathrm{Tr}_{\mathcal{S}}\left[\hat{M}_{m|n}\hat{\rho}_{\mathcal{S}}(0)\hat{M}_{m|n}^{\dagger}\right]} \,. \tag{32}$$

The first two terms in the sum are, respectively, the probabilities that $\mathcal{B}$ is initially in the state $|\chi_n\rangle$ and that it jumps from the state $|\chi_n\rangle$ to the state $|\chi_m\rangle$ in a time t. Their product is the probability that $\mathcal{B}$ is initially in the state $|\chi_n\rangle$ and in the state $|\chi_m\rangle$ at time t. The last term is then interpreted as the evolution of the density operator of the system $\mathcal{S}$ conditional to a jump of $\mathcal{B}$ from the state $|\chi_n\rangle$ to the state $|\chi_m\rangle$, see Fig. 1. It has exactly the same form as the unitary evolution of $\mathcal{S}$ for an isolated system where the evolution operator $\hat{U}_{\mathcal{S}}$ is replaced by the conditional Kraus operator $\hat{M}_{m|n}$, see Eq. (5). Note, however, that the Kraus operator being nonunitary, this evolution must be renormalized so as to preserve the unit trace of the density operator. We conclude that the Kraus operators $\hat{M}_{m|n}$ play the role of evolution operators for the system $\mathcal{S}$ conditioned to the jumps performed by $\mathcal{B}$. In other words, Eq. (32) is nothing but the average of the unitary-like evolutions of $\mathcal{S}$ averaged over the different state jumps of $\mathcal{B}$.

1.4 Kraus theorem

The general quantum map derived above is very useful for it provides us with a clear physical picture and an enlightening interpretation of the Kraus operators $\hat{M}_{m|n}$. As discussed above, these operators are constructed explicitly from all possible jumps of the system $\mathcal{B}$ from any state $|\chi_n\rangle$ to any state $|\chi_m\rangle$. In such a formulation, there are thus of

the order of $\dim(\mathcal{E}_\mathcal{B})^2$ Kraus operators, a priori all different[2]. If $\mathcal{B}$ is a very large system – as is most often the case when considering the coupling of the system of interest $\mathcal{S}$ to its whole environment – this represents a huge number of such operators. Yet, we must keep in mind that, in fine, the Kraus operators are used to describe the evolution of the system $\mathcal{S}$ and not of the system $\mathcal{B}$ to which it is coupled. Now, the former lives in a Hilbert space of dimension $\dim(\mathcal{E}_\mathcal{S})$, potentially much smaller than $\dim(\mathcal{E}_\mathcal{B})$. The density matrix of $\mathcal{S}$ is thus determined by of the order of $\dim(\mathcal{E}_\mathcal{S})^2$ independent coefficients. We may thus anticipate that only a set of about $\dim(\mathcal{E}_\mathcal{S})^2$ may be enough. It can indeed be shown that the Kraus operators are not unique and their number can be reduced to $\dim(\mathcal{E}_\mathcal{S})^2$.

Kraus theorem
Any completely positive map on a quantum system $\mathcal{S}$ may be written in the form

$$\hat{\rho}_\mathcal{S}(0) \longrightarrow \hat{\rho}_\mathcal{S}(t) = \mathcal{K}\big[\hat{\rho}_\mathcal{S}(0)\big] = \sum_{0\leq\alpha<\dim(\mathcal{E}_\mathcal{S})^2} \hat{M}_\alpha \hat{\rho}_\mathcal{S}(0) \hat{M}_\alpha^\dagger \tag{33}$$

with the completeness relation

$$\sum_\alpha \hat{M}_\alpha^\dagger \hat{M}_\alpha = \hat{\mathbb{1}}_\mathcal{S} \ . \tag{34}$$

The sum contains at most $\dim(\mathcal{E}_\mathcal{S})^2$ nonzero terms.

Hence, if for instance $\mathcal{S}$ is a qubit, it lives in a Hilbert space of dimension $\dim(\mathcal{E}_\mathcal{S}) = 2$ but it can be coupled to a much larger system. Think for instance about a two-level atom coupled to the radiation field, described by a continuous infinity of modes ℓ, each of which can accommodate an arbitrary number of photons $N_\ell \in \mathbb{N}$. While the discussion of Sec. 1.3 suggests we need an infinite number of Kraus operators, the Kraus theorem shows that one can find a formulation with only up to $\dim(\mathcal{E}_\mathcal{S})^2 = 4$ Kraus operators. This is a considerable simplification! It shows in particular that the construction of the Kraus operators should not be done naively. If one finds more than $\dim(\mathcal{E}_\mathcal{S})^2$, it indicates that further simplification is possible.

Interpretation of the compact Kraus form

The Kraus theorem has many important consequences.

On the one hand, it shows that the evolution of any open quantum system $\mathcal{S}$ can always be interpreted as resulting from the coupling to an effective (fictitious) environment $\mathcal{B}_{\text{eff}}$ of dimension $\dim(\mathcal{E}_{\mathcal{B}_{\text{eff}}}) = \dim(\mathcal{E}_\mathcal{S})^2$ at most and initially in a pure state, $\hat{\rho}_{\mathcal{B}_{\text{eff}}}(0) = |0\rangle\langle 0|$.

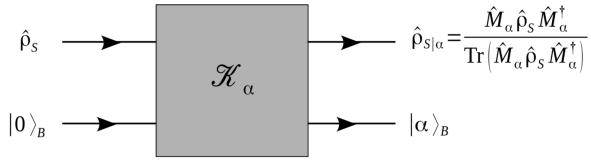
[2]Here we do not take into account the constraints imposed by the fact that the evolution operator of the bipartite system $\mathcal{S}\otimes\mathcal{B}$ is Hermitian.

The Kraus operators $\hat{M}_\alpha$ are then interpreted as the actions on $\mathcal{S}$ induced by a jump of $\mathcal{B}_{\text{eff}}$ from the state $|0\rangle$ to a state $|\alpha\rangle$. This interpretation is important especially when little is known about the properties of the real physical environment and/or of its interactions with the system of interest $\mathcal{S}$.

On the other hand, the quantum map

$$\hat{\rho}_{\mathcal{S}}(0) \longrightarrow \hat{\rho}_{\mathcal{S}}(t) = \mathcal{K}\big[\hat{\rho}_{\mathcal{S}}(0)\big] = \sum_\alpha \hat{M}_\alpha \hat{\rho}_{\mathcal{S}}(0) \hat{M}_\alpha^\dagger \tag{35}$$

has exactly the same form as that resulting from an unread generalized measurement. Hence, the Kraus operator $\hat{M}_\alpha$ describes the action of a (fictitious) measurement performed by the environment $\mathcal{B}$ on the system $\mathcal{S}$, having returned the result $|\alpha\rangle_{\mathcal{B}}$,



The quantum map (35) is then the average of all these processes, weighted by the probability of each jump in $\mathcal{B}$. In other words, the evolution of any open quantum system $\mathcal{S}$ governed by the coupling to another system $\mathcal{B}$ can be interpreted as the result of a set of unread measurements performed on the system $\mathcal{S}$ by the system $\mathcal{B}$. This is perfectly consistent since the description we made in the context of POVMs. There, we described the interaction of a system $\mathcal{S}$ with the meter M, which played exactly the same role as the system $\mathcal{B}$ here. If, as done here, we do not use it as a meter, we do not get any information about the measurement and the evolution is that of an unread measurement.

2 System coupled to a bath: The Lindblad equation

So far we have considered the influence of the coupling to an arbitrary system $\mathcal{B}$ on the dynamics of the system of interest $\mathcal{S}$, and we have found the most general form of its evolution, as a Kraus map. We now make hypotheses on the system $\mathcal{B}$ and assume it is a bath in the thermodynamic sense. As we shall see, this will allow us to find a differential equation for the dynamics of $\mathcal{S}$, in the form of the so-called *Lindblad equation*.

2.1 Formal derivation of the Lindblad equation

To find a differential evolution equation, write first the Kraus quantum map (33) as a stroboscopic dynamical process,

$$\hat{\rho}_{\mathcal{S}}(0) \longrightarrow \hat{\rho}_{\mathcal{S}}(\Delta t) \longrightarrow \hat{\rho}_{\mathcal{S}}(2\Delta t) \longrightarrow \ldots \longrightarrow \hat{\rho}_{\mathcal{S}}(t) \longrightarrow \hat{\rho}_{\mathcal{S}}(t+\Delta t) \longrightarrow \ldots \quad (36)$$

with

$$\hat{\rho}_{\mathcal{S}}(t+\Delta t) = \sum_{\alpha} \hat{M}_{\alpha}(t,\Delta t)\hat{\rho}_{\mathcal{S}}(t)\hat{M}_{\alpha}(t,\Delta t)^{\dagger} \ , \quad (37)$$

where the set of $\hat{M}_{\alpha}(t,\Delta t)$ spans the Kraus operators describing the evolution of $\mathcal{S}$ between the times t and $t+\Delta t$. According to the discussion of Sec. 1.4, the dynamics of the system $\mathcal{S}$ is equivalent to a series of unread measurements. Due to the linearity of the map with respect to the density operator, this may alternatively be seen as a series of readout measurements, then averaged over the measurement results. Each measurement generates a state jump in $\mathcal{S}$ induced by a state jump in $\mathcal{B}$, while between two jumps, the system $\mathcal{S}$ evolves according to its own dynamics governed by a Hamiltonian $\hat{H}$, see Fig. 2(a). In principle, the Kraus operators $\hat{M}_{\alpha}(t,\Delta t)$ depend not only on the time step Δt but also on the time t at which they apply. This is because they depend on the state of the bath $\mathcal{B}$, which evolves together with the system $\mathcal{S}$. Hence, in principle, the Kraus operators $\hat{M}_{\alpha}(t,\Delta t)$ depend on the complete history of $\mathcal{S}\otimes\mathcal{B}$ between the times 0 and t.

Assume now that the system $\mathcal{B}$ is a *bath*. According to the thermodynamic definition, it is a huge system – formally infinitely larger than the system $\mathcal{S}$ to which it is coupled. In this case, the bath $\mathcal{B}$ may strongly affect the system $\mathcal{S}$ but, in turn, $\mathcal{S}$ only marginally affects $\mathcal{B}$. Hence, if we assume that the bath is initially in an equilibrium state, we can assume that it approximately remains at equilibrium in spite of its interactions with $\mathcal{S}$. More precisely, consider for instance the interaction of an atom with the radiation field in the vacuum. A typical jump corresponds to the emission of a spontaneous photon, which then populates a particular mode ℓ_1. After such an event, the radiation field is not in the vacuum any longer but in a state where all modes are empty, except the mode ℓ_1, which contains a single photon. In an infinite system, however, the spontaneous emission is induced by the interaction of the atom with a continuous infinity of modes ℓ (all those whose angular frequency ω_ℓ equals that of the atomic transition ω_A up to the bandwidth Γ). The modification of the population of only one of these modes has therefore a negligible impact on the subsequent spontaneous emission of an excited atom. Moreover, the bath being very large, it strongly interacts with its environment and its state will relax very quickly towards the initial equilibrium state, in a typical time $\tau_{\mathcal{B}}^{*}$, see blue bubble in Fig. 2(a). In the example above, the photon emitted in mode ℓ_1 may be reabsorbed by the walls of the cell where the atoms are trapped. For an atom in a $d = 10\,\mathrm{cm}$ wide cell, a spontaneous photon reaches the walls in typically $\tau \simeq d/2c$ where $c \simeq 3\times10^{8}\,\mathrm{m\cdot s^{-1}}$ is the speed of light, i.e. $\tau \simeq 0.15\,\mathrm{ns}$. This is much shorter than the typical decay time of a atomic excited state, e.g. $\Gamma^{-1} \simeq 27\,\mathrm{ns}$ for rubidium atoms.

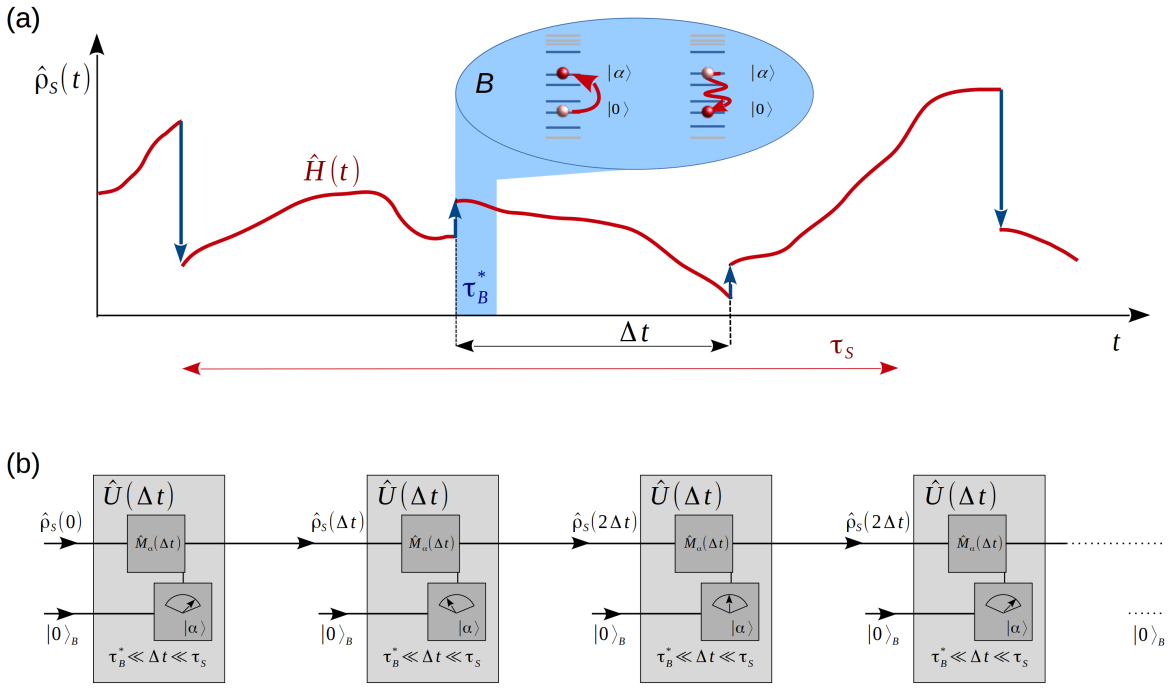


Figure 2: Dynamics of an open quantum system $\mathcal{S}$ coupled to a bath $\mathcal{B}$. (a) Schematic representation of the evolution of the density matrix of $\mathcal{S}$. The system undergoes a Hamiltonian evolution governed by its effective Hamiltonian $\hat{H}(t)$, interrupted by jumps induced by jumps in the bath. The bath relaxes rapidly to its equilibrium state, in a time $\tau_{\mathcal{B}}^*$ much shorter than the typical evolution time $\tau_{\mathcal{S}}$ of the system. (b) Representation in the form of a stroboscopic circuit. At each elementary time step Δt, the system undergoes an Hamiltonian evolution followed by a measurement, the whole being described by the Kraus operator $\hat{M}_\alpha(\Delta t)$ where $|\alpha\rangle$ is the measurement result imprinted in $\mathcal{B}$. The state of the bath, which plays the role of the measuring apparatus, is reset before each measurement in the reference state $|0\rangle$. The evolution of $\mathcal{S}$ is the average of these trajectories over all measurement results.

More generally, if we assume that the relaxation time in the bath, $\tau_{\mathcal{B}}^*$, is much shorter than the dynamical time scale of the system of interest, $\tau_{\mathcal{S}}$, the influence of $\mathcal{B}$ on the dynamics of $\mathcal{S}$, and thus the Kraus operators, are almost independent of time, and we may write

$$\hat{\rho}_{\mathcal{S}}(t+\Delta t) \simeq \sum_\alpha \hat{M}_\alpha(\Delta t)\hat{\rho}_{\mathcal{S}}(t)\hat{M}_\alpha(\Delta t)^\dagger \ , \tag{38}$$

where the Kraus operators $\hat{M}_\alpha(\Delta t)$ stills depend on the time step Δt but not any longer on the time t at which they apply. The stroboscopic dynamics of the system $\mathcal{S}$ may now be represented as a series of independent measurements, the bath (meter) being systematically reinitialized in the same state before any measure, see Fig. 2(b).

As discussed above, Eq. (38) is valid when the time step Δt is much longer that the bath relaxation time $\tau_{\mathcal{B}}^*$. Assuming that this time step Δt is at the same time much shorter than the dynamical time scale of the system of interest $\tau_{\mathcal{S}}$, the density operator of the latter evolves very little over each time step and we may write

$$\hat{\rho}_{\mathcal{S}}(t+\Delta t) = \hat{\rho}_{\mathcal{S}}(t) + \mathrm{O}\left(\Delta t\right) \ . \tag{39}$$

According to the discussion above, this formulation is valid under the hierarchy of time scales

$$\tau_{\mathcal{B}}^* \ll \Delta t \ll \tau_{\mathcal{S}} \ . \tag{40}$$

The finite time step evolution is now to be determined from Eq. (38) and the corresponding Kraus operators. To do so, we may take advantage of the nonunique form of the Kraus operators and group together all the contributions close to unity in one of them, say $\hat{M}_0(\Delta t)$. We may then write

$$\begin{cases} \hat{M}_0(\Delta t) = \hat{\mathbb{1}} + \left(\hat{A} - \frac{i}{\hbar}\hat{H}\right)\Delta t + \mathrm{o}\left(\Delta t\right) \\ \\ \hat{M}_{\alpha\geq 1}(\Delta t) = \hat{L}_\alpha \sqrt{\Delta t} + \mathrm{o}\left(\sqrt{\Delta t}\right) \end{cases} , \tag{41}$$

where we have expanded the first-order correction to unity of $\hat{M}_0(\Delta t)$ into a Hermitian part and an anti-Hermitian part, $\hat{M}_0 - \hat{\mathbb{1}} = (\hat{A} - \frac{i}{\hbar}\hat{H})\Delta t$, where both $\hat{A}$ and $\hat{H}$ are Hermitian operators. Since the operators $\hat{M}_{\alpha\geq 1}(\Delta t)$ have a quadratic contribution, they must be proportional to $\sqrt{\Delta t}$. The so-called *Lindblad operators* $\hat{L}_\alpha$, just as the Kraus operators $\hat{M}_\alpha$, are in general non-Hermitian. The operator $\hat{A}$ may be found using the completeness relation of the Kraus operators. Inserting the formulas of Eq. (41) into Eq. (34), we find

$$\hat{\mathbb{1}} = \hat{\mathbb{1}} + \left(\hat{A} + \frac{i}{\hbar}\hat{H} + \hat{A} - \frac{i}{\hbar}\hat{H}\right)\Delta t + \sum_{\alpha\geq 1} \hat{L}_\alpha^\dagger \hat{L}_\alpha \Delta t + \mathrm{o}(\Delta t) \ , \tag{42}$$

which yields

$$\hat{A} = -\frac{1}{2}\sum_{\alpha\geq 1} \hat{L}_\alpha^\dagger \hat{L}_\alpha \ . \tag{43}$$

Finally, combining Eqs. (38) and (41), we find

$$\hat{\rho}_{\mathcal{S}}(t+\Delta t) = \hat{\rho}_{\mathcal{S}}(t) + \left(\hat{A} - \frac{i}{\hbar}\hat{H}\right)\Delta t \hat{\rho}_{\mathcal{S}}(t) + \hat{\rho}_{\mathcal{S}}(t)\left(\hat{A} + \frac{i}{\hbar}\hat{H}\right)\Delta t + \sum_{\alpha\geq 1} \hat{L}_\alpha \hat{\rho}_{\mathcal{S}}(t) \hat{L}_\alpha^\dagger \, \Delta t + \mathrm{o}(\Delta t) \ . \tag{44}$$

Writing $\Delta\hat{\rho}_{\mathcal{S}} = \hat{\rho}_{\mathcal{S}}(t+\Delta t) - \hat{\rho}_{\mathcal{S}}(t)$, we then find

$$\frac{\Delta\hat{\rho}_{\mathcal{S}}}{\Delta t} = \hat{A}\hat{\rho}_{\mathcal{S}} + \hat{\rho}_{\mathcal{S}}\hat{A} - \frac{i}{\hbar}\hat{H}\hat{\rho}_{\mathcal{S}} + \frac{i}{\hbar}\hat{\rho}_{\mathcal{S}}\hat{H} + \sum_{\alpha\geq 1} \hat{L}_\alpha \hat{\rho}_{\mathcal{S}} \hat{L}_\alpha^\dagger \ . \tag{45}$$

Finally, using Eq. (43) and replacing the finite difference ratio by a *coarse-grained derivative*, we find a closed equation for the density matrix of $\mathcal{S}$:

Quantum master equation in the Lindblad form
The time evolution equation of the density matrix of any quantum system $\mathcal{S}$ coupled to a bath reads as

$$\frac{d\hat{\rho}_{\mathcal{S}}}{dt} = \frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}_{\mathcal{S}}\right] + \sum_{\alpha\geq 1}\left(\hat{L}_\alpha \hat{\rho}_{\mathcal{S}} \hat{L}_\alpha^\dagger - \frac{1}{2}\hat{\rho}_{\mathcal{S}}\hat{L}_\alpha^\dagger \hat{L}_\alpha - \frac{1}{2}\hat{L}_\alpha^\dagger \hat{L}_\alpha \hat{\rho}_{\mathcal{S}}\right) , \tag{46}$$

where there are at most $\dim\left(\mathcal{E}_{\mathcal{S}}\right)^2 - 1$ Lindblad operators $\hat{L}_\alpha$.

The Lindblad equation is also called the *master equation*. It is the most general form of the Markovian evolution of an open quantum system and makes no special assumptions about the coupling between the system and the bath. The only important assumption is that the bath is vanishingly perturbed by its interaction with the system and that it relaxes rapidly towards its equilibrium state.

2.2 Physical content of the Lindblad equation

The Lindblad equation contains two terms that play radically different roles.

On the one hand, the first term in the right-hand side of Eq. (46), $\frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}_{\mathcal{S}}\right]$, is similar to the evolution of an isolated system. If alone, it thus describes a unitary evolution, which in particular conserves the purity, as well as any entanglement entropy. In other words, a system in a pure state remains in a pure state and a system in a mixed state remains in a mixed state. This term is also nondissipative so that it conserves the energy of the system. Note, however, that the effective Hamiltonian $\hat{H}$ appearing in Eq. (46) generally differs from the bare Hamiltonian of the system $\mathcal{S}$, $\hat{H}_{\mathcal{S}}$. Nevertheless, when the coupling to the bath is weak, we expect $\hat{H} \simeq \hat{H}_{\mathcal{S}}$..

On the other hand, the second term in the right-hand side of Eq. (46),

$$\mathcal{L}'[\hat{\rho}_{\mathcal{S}}] = \sum_{\alpha\geq 1}\left(\hat{L}_\alpha \hat{\rho}_{\mathcal{S}} \hat{L}_\alpha^\dagger - \frac{1}{2}\hat{\rho}_{\mathcal{S}}\hat{L}_\alpha^\dagger \hat{L}_\alpha - \frac{1}{2}\hat{L}_\alpha^\dagger \hat{L}_\alpha \hat{\rho}_{\mathcal{S}}\right) , \tag{47}$$

is called a *Liouvillian*. It describes a nonunitary evolution of the system $\mathcal{S}$ induced by jumps in the bath $\mathcal{B}$. The latter generate jumps in the system $\mathcal{S}$ similar to (unread) measurement processes. They are described by the Lindblad operators $\hat{L}_\alpha$, which may alternatively be called *quantum jump operators*. In the most general case, this term does not preserve the state purity so that a pure state can be transformed into a mixed state and vice versa. There is no general rule and the entropy of the system may increase or decrease depending on the situation. This term is also dissipative and describes energy exchanges between the system and the bath. The energy of the system is therefore generally not conserved.

Note that the prime sign is used to stress that the Liouvillian $\mathcal{L}'[\hat{\rho}_\mathcal{S}]$ written above refers to the part of the evolution associated to the Lindblad terms. In contrast, the total Liouvillian,

$$\mathcal{L}[\hat{\rho}_\mathcal{S}] = \frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}_\mathcal{S}\right] + \sum_{\alpha\geq 1}\left(\hat{L}_\alpha \hat{\rho}_\mathcal{S} \hat{L}_\alpha^\dagger - \frac{1}{2}\hat{\rho}_\mathcal{S}\hat{L}_\alpha^\dagger \hat{L}_\alpha - \frac{1}{2}\hat{L}_\alpha^\dagger \hat{L}_\alpha \hat{\rho}_\mathcal{S}\right) \ , \tag{48}$$

describes the complete evolution of the density matrix of $\mathcal{S}$, including both the unitary and the nonunitary contributions.

3 Usecases

We now apply the Lindblad formalism to a few paradigmatic physical situations. The exact derivation of the Kraus operators is a tedious task, which will be tackled in Chap. **??** via the *Born-Markov approach*. Here we only rely on simple physical arguments, which nevertheless turn out to be sufficient to write the exact forms of these operators. The two examples below deal, on the one hand, with the relaxation of an open system towards thermodynamic equilibrium and, on the other hand, with the dynamics of a driven open system.

3.1 Relaxation towards equilibrium: The damped harmonic oscillator

Consider first the relaxation dynamics of an open quantum system. For the sake of simplicity, consider a harmonic oscillator coupled to a bath of harmonic oscillators.

Model

More precisely, the system of interest $\mathcal{S}$ is a single 1D harmonic oscillator, governed by the Hamiltonian

$$\hat{H}_\mathcal{S} = \hbar\omega_\mathcal{S}(\hat{c}^\dagger\hat{c} + 1/2) \ , \tag{49}$$

where $\omega_\mathcal{S}$ is the system's angular frequency and $\hat{c}$ the lowering operator, with the commutation relation $[\hat{c}, \hat{c}^\dagger] = 1$. The bath is made of an infinite number of 1D harmonic oscillators and is governed by the Hamiltonian

$$\hat{H}_\mathcal{B} = \sum_\ell \hbar\omega_\ell(\hat{b}_\ell^\dagger\hat{b}_\ell + 1/2) \ . \tag{50}$$

Each oscillator corresponds to an excitation mode ℓ of the bath, with angular frequency ω_ℓ, and $\hat{b}_\ell$ is the annihilation operator of an excitation quantum in mode ℓ. These operators

fulfill the commutation relations $[\hat{b}_\ell, \hat{b}_{\ell'}^\dagger] = \delta_{\ell,\ell'}$. We further assume that the system is coupled to each bath mode, via the interaction Hamiltonian

$$\hat{H}_\mathrm{I} = \sum_\ell \left(\hbar\kappa_\ell \hat{b}_\ell \hat{c}^\dagger + \hbar\kappa_\ell^* \hat{b}_\ell^\dagger \hat{c} \right) . \tag{51}$$

This coupling term describes correlated jumps in the system and either mode of the bath, each with a coupling strength κ_ℓ: The first term under the sum describes a jump of the system to a higher energy state associated with the annihilation of an energy quantum in the mode ℓ of the bath, thereby transferring energy from the bath to the system. Conversely, the second term describes a jump of the system to a lower energy state associated with the creation of an energy quantum in the mode ℓ of the bath, thereby transferring energy from the system to the bath.

A concrete application example of this model can be realized in cavity quantum electrodynamics: The system $\mathcal{S}$ is the cavity mode and the bath $\mathcal{B}$ is constituted by the vibration modes of the mirrors.

Lindblad equation

The general form of the Kraus and Lindblad operators results directly from the elementary jumps in the bath: On the one hand, the absorption by the system of an excitation quantum of the bath from the mode ℓ corresponds to a jump in the excitation number such as $N_\ell \to N_\ell - 1$. This is induced by the term $\hat{b}_\ell \hat{c}^\dagger$ in the coupling Hamiltonian and produces a Kraus operator such as $\hat{M}_{N_\ell \to N_\ell - 1} \propto \sqrt{N_\ell}\, \hat{c}^\dagger$, where the term $\sqrt{N_\ell}$ is nothing but bosonic amplification. Since they are all proportional to $\hat{c}^\dagger$, they induce by summation over the modes a single Lindblad operator, which we write

$$\hat{L}_+ = \sqrt{\Gamma'}\, \hat{c}^\dagger . \tag{52}$$

On the other hand, the creation of an excitation in the mode ℓ corresponds to an opposite jump such as $N_\ell \to N_\ell + 1$. This one is induced by the term in $\hat{b}_\ell^\dagger \hat{c}$ and produces a Kraus operator such as $\hat{M}_{N_\ell \to N_\ell + 1} \propto \sqrt{N_\ell + 1}\, \hat{c}$. They induce by summation over the modes a single Lindblad operator, which we write

$$\hat{L}_- = \sqrt{\Gamma + \Gamma'}\, \hat{c} . \tag{53}$$

The prefactors of the Lindblad operators are written $\sqrt{\Gamma'}$ and $\sqrt{\Gamma + \Gamma'}$ for convenience. The physical meanings of the quantities Γ and Γ' are unveiled below. This discussion yields the Lindblad equation

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}\left[\hbar\omega_0\, \hat{c}^\dagger \hat{c}, \hat{\rho}\right] + (\Gamma + \Gamma') \left(\hat{c}\hat{\rho}\hat{c}^\dagger - \frac{1}{2}\hat{\rho}\hat{c}^\dagger\hat{c} - \frac{1}{2}\hat{c}^\dagger\hat{c}\hat{\rho} \right) + \Gamma' \left(\hat{c}^\dagger\hat{\rho}\hat{c} - \frac{1}{2}\hat{\rho}\hat{c}\hat{c}^\dagger - \frac{1}{2}\hat{c}\hat{c}^\dagger\hat{\rho} \right)} . \tag{54}$$

While the general form of this Lindblad equation has been written from simple arguments, it can be shown to be exact and the expressions of the coefficients Γ and Γ' can be found

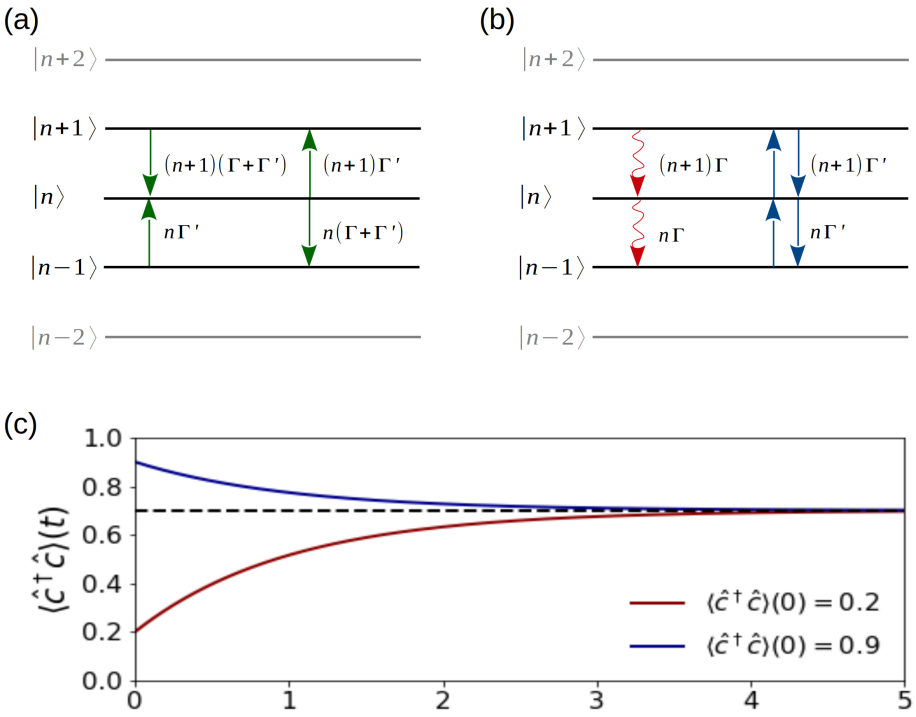


Figure 3: Relaxation of a 1D harmonic oscillator coupled to a bath. (a) and (b) show elementary population transfer processes. (c) Relaxation of the average excitation number $\langle \hat{c}^\dagger \hat{c} \rangle(t)$, Eq. (60), for $\Gamma'/\Gamma = 0.7$ and two different initial values.

explicitely. Note that the effective frequency of the oscillator, ω_0, may differ from the bare frequency $\omega_{\mathcal{S}}$ owing to non-state changing processes induced by the bath. The latter are, however, often very weak and $\omega_0 \simeq \omega_{\mathcal{S}}$.

Steady state

We now discuss the solution of the Lindblad equation, and focus on the population of each harmonic-oscillator state $|n\rangle$ of the system. The latter is given by $p_n = \langle n| \hat{\rho} |n\rangle$, that is the probability that the system described by the density matrix $\hat{\rho}$ is in the state $|n\rangle$. Its dynamics is given by inserting, respectively, the bra $\langle n|$ and the ket $|n\rangle$ on the left and the right of the Lindblad equation (54). It yields

$$\frac{dp_n}{dt} = (\Gamma + \Gamma')\Big[(n+1)p_{n+1} - np_n\Big] + \Gamma'\Big[np_{n-1} - (n+1)p_n\Big] \ . \tag{55}$$

Remarkably, we find that the coherence terms $\langle n| \hat{\rho} |m\rangle$ with $n \neq m$ do not contribute. We hence obtain a pure classical stochastic rate equation. On the one hand, the terms in $(n+1)p_{n+1}$ and np_{n-1} correspond to filling processes of the state $|n\rangle$ from the states $|n+1\rangle$ and $|n-1\rangle$ at the rates $(n+1)(\Gamma+\Gamma')$ and $n\Gamma'$, respectively. On the other hand, the terms in $-np_n$ and $-(n+1)p_n$ correspond to leaking processes from the state $|n\rangle$ towards the states $|n-1\rangle$ and $|n+1\rangle$ at the rates $n(\Gamma+\Gamma')$ and $(n+1)\Gamma'$, respectively, see Fig. 3(a). The same processes can be reinterpreted by separating the contributions associated with Γ on the one hand and Γ' on the other hand, see Fig. 3(b). The terms

associated to Γ are population transfer processes from the state $|n\rangle$ to the state $|n-1\rangle$ and are assimilable to spontaneous emission: The system loses energy by successively creating excitations in the different modes of the bath. Note that these terms exist even when the bath is in its ground state, $\langle N_\ell \rangle = 0$ for all modes ℓ, since they are generated by Kraus operators of the form $\hat{M}_{N_\ell \to N_\ell + 1} \propto \sqrt{N_\ell + 1}\hat{c} \neq 0$ for $N_\ell = 0$. The terms associated to Γ' are then assimilated to stimulated emission and absorption processes. They induce transitions between the states $|n\rangle$ and $|n-1\rangle$ with the same rates in either direction, and exist only if the bath contains excitations, $\langle N_\ell \rangle \neq 0$.

The dynamics clearly yields a relaxation. The steady state is given by setting $dp_n/dt = 0$, which yields

$$\frac{p_{n+1}}{p_n} = \frac{\Gamma'}{\Gamma + \Gamma'} \ . \tag{56}$$

This formula may be derived by iteration from $n = 0$. Since the system is a 1D harmonic oscillator, its eigenenergies are equidistant, $E_n \simeq \hbar\omega_{\mathcal{S}}(n + 1/2)$, and the geometric decay of the populations, Eq. (56), may be written as

$$p_n = \frac{1}{Z} \exp\left(-E_n/k_{\mathrm{B}}T\right) \qquad \text{with} \quad Z = \sum_n \exp\left(-E_n/k_{\mathrm{B}}T\right) \ . \tag{57}$$

The quantity Z is the probability normalization factor and T is such that $\Gamma'/(\Gamma + \Gamma') = \exp\left(-\hbar\omega_{\mathcal{S}}/k_{\mathrm{B}}T\right)$. We thus find that the steady state of the system $\mathcal{S}$ is nothing but a thermodynamic equilibrium state at the effective temperature T. Note that since $0 \leq \Gamma'/(\Gamma+\Gamma') \leq 1$, this temperature is a well-defined nonnegative quantity. In fact, it may be shown using the formalism of lecture 7 that T is the temperature of the bath. Therefore, we have obtained nothing but a thermodynamic equilibrium of the system and the bath, where the bath imposes its temperature to the system. This is consistent with standard thermodynamics.

Relaxation towards equilibrium

The Lindblad equation also gives the relaxation dynamics towards equilibrium. Consider for instance the relaxation of energy, $U = \hbar\omega_{\mathcal{S}}(\langle \hat{c}^\dagger \hat{c} \rangle + 1/2)$. We obtain its dynamical equation by writing $\langle \hat{c}^\dagger \hat{c} \rangle = \mathrm{Tr}\left(\hat{c}^\dagger \hat{c}\right)$. Using the Lindblad equation, we find

$$\frac{d\langle \hat{c}^\dagger \hat{c} \rangle}{dt} = -\Gamma \langle \hat{c}^\dagger \hat{c} \rangle + \Gamma' \ . \tag{58}$$

It follows that the quantity $\langle \hat{c}^\dagger \hat{c} \rangle$ relaxes towards an equilibrium state given by

$$\langle \hat{c}^\dagger \hat{c} \rangle_{\mathrm{eq}} = \frac{\Gamma'}{\Gamma} = \frac{1}{\exp\left(\hbar\omega_{\mathcal{S}}/k_{\mathrm{B}}T\right) - 1} \ , \tag{59}$$

where we have used the definition of the temperature T given above. This is nothing but the Bose-Einstein formula, corresponding to the average excitation number of a 1D

harmonic oscillator. Moreover, the relaxation rate is Γ and does not depend on Γ'. This is natural since the stimulated processes happen at the same rate from $|n\rangle$ to $|n-1\rangle$ and vice versa. More precisely, given the initial state, the explicit solution of the dynamical equation (58) reads as

$$\langle \hat{c}^\dagger \hat{c} \rangle(t) = \frac{\Gamma'}{\Gamma} \left(1 - \mathrm{e}^{-\Gamma t}\right) + \mathrm{e}^{-\Gamma t} \langle \hat{c}^\dagger \hat{c} \rangle(0) \ . \tag{60}$$

This solution is shown on Fig. 3(c) for two initial conditions depending on whether the initial value $\langle \hat{c}^\dagger \hat{c} \rangle(0)$ is higher or lower than the equilibrium value Γ'/Γ.

3.2 Driven open system: Optical Bloch equations

Consider now an open system subjected to an oscillating external field. A typical example is that of an atom subjected to a laser field.

The atom is considered as a two-level system, with ground state $|\mathrm{g}\rangle$ and excited state $|\mathrm{e}\rangle$, separated by a transition energy $E_\mathrm{e} - E_\mathrm{g} = \hbar\omega_A$. We call $\hat{\sigma}_+ = |\mathrm{e}\rangle\langle\mathrm{g}|$ and $\hat{\sigma}_- = |\mathrm{g}\rangle\langle\mathrm{e}|$ the atomic raising and lowering operators. The atom is subjected to a monochromatic laser field of angular frequency ω_L. The atom-field interaction, described within electric dipole approximation, reads as

$$\hat{V}_\mathrm{AL}(\boldsymbol{R}, t) = \hat{\sigma}_+ \frac{\hbar\Omega_\mathrm{L}(\boldsymbol{R})}{2} \mathrm{e}^{-i\omega_\mathrm{L} t} + \hat{\sigma}_- \frac{\hbar\Omega_\mathrm{L}(\boldsymbol{R})^*}{2} \mathrm{e}^{+i\omega_\mathrm{L} t} \ , \tag{61}$$

where Ω_L is the Rabi angular frequency and $\boldsymbol{R}$ is the position of the atom.

The radiation field is assumed to be in thermodynamic equilibrium at temperature T. The average photon number in any mode ℓ is given by the Bose-Einstein distribution

$$\langle N_\ell \rangle = \frac{1}{\exp\left(\hbar\omega_\ell / k_\mathrm{B} T\right) - 1} \ . \tag{62}$$

Since we are interested in the coupling to the atom, the relevant modes are those with angular frequency $\omega_\ell \simeq \omega_A$. In the optical domain, corresponding to wavelengths $\lambda \simeq 400 - 700\,\mathrm{nm}$, we obtain $\hbar\omega_\ell / k_\mathrm{B} \simeq 2 - 4 \times 10^4\,\mathrm{K}^{-1}$. At room temperature, $T \simeq 300\,\mathrm{K}$, we obtain $\hbar\omega_\ell / k_\mathrm{B} T \simeq 60 - 120$ and $\langle N_\ell \rangle \simeq 10^{-26} - 10^{-52}$. To an excellent approximation, the relevant modes of the radiation field are thus essentially empty at room temperature and the field can be considered in its ground state. This still holds up to temperatures of the order of a few thousand Kelvins.

The coupling between the atom and the radiation field is considered also in the electric dipole approximation and reads as

$$\hat{H}_\mathrm{I} = \sum_\ell \left(\hbar\kappa_\ell \hat{a}_\ell \hat{\sigma}_+ + \hbar\kappa_\ell^* \hat{a}_\ell^\dagger \hat{\sigma}_- \right) \ , \tag{63}$$

where $\hat{a}_\ell$ in the photon annihilation operator in mode ℓ. This coupling has the same form as in the damped harmonic oscillator problem discussed in the Sec. 3.1. Therefore, it generates similar jump terms. However, the field being in vacuum, only the spontaneous emission terms remain and there is a unique Linblad operator of the form

$$\hat{L}_- = \sqrt{\Gamma}\,\hat{\sigma}_- \ . \tag{64}$$

Including the coupling with the laser field, which is unitary in contrast to the coupling to the quantized vacuum, and using the relations $\hat{\sigma}_+ = \hat{\sigma}_-^\dagger$ and $\hat{\sigma}_+\hat{\sigma}_- = |\mathrm{e}\rangle\langle\mathrm{e}|$, we obtain the Lindblad equation

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}\Big[\hbar\omega_0\,|\mathrm{e}\rangle\langle\mathrm{e}| + \hat{V}_{\mathrm{AL}}(t), \hat{\rho}\Big] + \Gamma\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\hat{\rho}\,|\mathrm{e}\rangle\langle\mathrm{e}| - \frac{1}{2}\,|\mathrm{e}\rangle\langle\mathrm{e}|\,\hat{\rho}\right)}\ , \tag{65}$$

where $\hat{H}_0 = \hbar\omega_0\,|\mathrm{e}\rangle\langle\mathrm{e}|$ is the atomic Hamiltonian, with an angular transition frequency $\omega_0 \simeq \omega_A$, slightly modified by the Lamb shift.

This formula is equivalent to the standard optical Bloch equations (OBE). To show it, apply the bra and ket corresponding to the atom ground and excited states, respectively on the left-hand and right-hand sides of the Lindblad equation (65). It yields

$$\frac{d\rho_{\mathrm{ee}}}{dt} = -\Gamma\rho_{\mathrm{ee}} + i\frac{\Omega_{\mathrm{L}}^*}{2}\mathrm{e}^{+i\omega_{\mathrm{L}}t}\rho_{\mathrm{eg}} - i\frac{\Omega_{\mathrm{L}}}{2}\mathrm{e}^{-i\omega_{\mathrm{L}}t}\rho_{\mathrm{ge}} \tag{66}$$

$$\frac{d\rho_{\mathrm{gg}}}{dt} = +\Gamma\rho_{\mathrm{ee}} - i\frac{\Omega_{\mathrm{L}}^*}{2}\mathrm{e}^{+i\omega_{\mathrm{L}}t}\rho_{\mathrm{eg}} + i\frac{\Omega_{\mathrm{L}}}{2}\mathrm{e}^{-i\omega_{\mathrm{L}}t}\rho_{\mathrm{ge}} \tag{67}$$

$$\frac{d\rho_{\mathrm{eg}}}{dt} = -\left(\frac{\Gamma}{2} + i\omega_0\right)\rho_{\mathrm{eg}} - i\frac{\Omega_{\mathrm{L}}}{2}\mathrm{e}^{-i\omega_{\mathrm{L}}t}\left(\rho_{\mathrm{ee}} - \rho_{\mathrm{gg}}\right) \tag{68}$$

$$\frac{d\rho_{\mathrm{ge}}}{dt} = -\left(\frac{\Gamma}{2} - i\omega_0\right)\rho_{\mathrm{ge}} + i\frac{\Omega_{\mathrm{L}}^*}{2}\mathrm{e}^{+i\omega_{\mathrm{L}}t}\left(\rho_{\mathrm{ee}} - \rho_{\mathrm{gg}}\right)\ . \tag{69}$$

The first two equations deal with the populations of the excited and ground states, respectively. They are consistent with the normalization condition $\rho_{\mathrm{gg}} + \rho_{\mathrm{ee}} = 1$. The last two are consistent with the fact that the density matrix is Hermitian, $\rho_{\mathrm{eg}}^* = \rho_{\mathrm{ge}}$. In contrast to the damped harmonic oscillator discussed in Sec. 3.1, the evolution of the populations of the ground and excited states, ρ_{gg} and ρ_{ee}, depend on the coherence terms, ρ_{ge} and ρ_{ge}, and vice versa. Hence, we do not obtain classical rate equations here. This coupling between populations and coherences is induced by the driving, i.e. the coupling of the atom to the time-dependent laser field $\hat{V}_{\mathrm{AL}}(\boldsymbol{R}, t)$, which was absent in the case of the damped harmonic oscillator. In fact, although the interaction Hamiltonian $\hat{V}_{\mathrm{AL}}(\boldsymbol{R}, t)$ and the Lindblad operators $\hat{L}_\pm$ induce similar elementary processes, they lead to radically different dynamics. Indeed, they both induce atomic state transitions generated by the operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$. However, the interaction Hamiltonian $\hat{V}_{\mathrm{AL}}(\boldsymbol{R}, t)$ is a coherent coupling term, which typically produces Rabi oscillations. Conversely, the Lindblad operators $\hat{L}_\pm$ are incoherent and lead to dissipative dynamics, i.e. typically exponential

relaxation towards a steady state. When these two terms are present, as in the general case considered here, the solution of the optical Bloch equations are damped oscillations, see part 3 of the 2021-2022 exam.