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Lecture 5: Quantum measurement

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Along with entanglement, measurement processes are one of the most fundamentally quantum aspects of the theory, and profoundly change our understanding of the physical world. Although we have at our disposal a robust operational formalism to efficiently describe measurement processes, their interpretation and profund understanding are still subject to debates and different interpretations remain on the table. This concerns in particular the role of the observer in measurement processess. For a detailed discussion, see for instance Ref. [1]. Here, we shall not discuss these questions, which are still rather a matter of philosophical debate than a physical question. Instead, we shall focus on a concrete discussion of measurement processes, at the heart of physical predictions.

Measurement processes in quantum physics are primarily associated with the notions of probability and collapse of the wave function. This is one of the most puzzling aspects of quantum physics. Contrary to what happens in classical physics, a quantum measurement does not simply extract information about the state of the system. It forces the system to acquire the value obtained by the measurement. Thus, if a particle was in a well-defined momentum state, its position is totally undetermined. But, by performing a position measurement, i.e. by asking the particle where it is, we obtain a well-defined value and we force the particle to acquire this position. Its momentum then becomes totally undetermined.

As we shall see, this is, however, only one aspect of the problem. More fundamentally, measurement theory is associated with a large number of subtle concepts, including projection valued measurements (PVM), positive operator valued measurements (POVM), quantum non demolition (QND) measurements, ... All these aspects are discussed in this chapter. Before proceeding, it is worth recalling the obvious: A measurement in quantum mechanics, as anywhere else, primarily serves to acquire information about the state of the system.

1 Projective measurements (PVM)

Let us start by describing standard measurements in quantum physics, namely projective measurement processes. Technically, one usually speaks of projection-valued measurements (PVM), a term inherited from a very general underlying mathematical theory that we shall not try to explain in detail here. Let us simply keep in mind that the result of

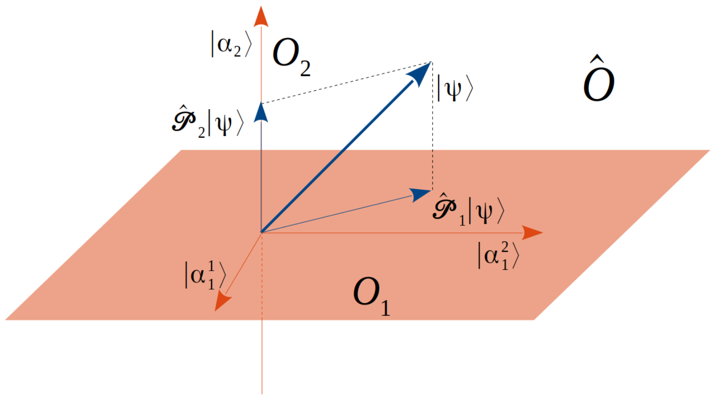

Figure 1: Measurement theory. When we measure an observable $\hat{\mathcal{O}}$ on a system described by the ket $|\psi\rangle$, one can find any of the eigenvalues O_j of $\hat{\mathcal{O}}$. The probability to find the value O_j is the square modulus of the projection of the ket onto the corresponding eigenspace, $|\hat{\mathcal{P}}_j|\psi\rangle|^2$, generated by the kets $|\alpha_j^\nu\rangle$, where ν accounts for the possible degeneracy of O_j. If O_j is found, the ket is then projected onto the corresponding eigenspace, and renormalized, $|\psi\rangle \to |\psi'\rangle = \hat{\mathcal{P}}_j|\psi\rangle / \big|\hat{\mathcal{P}}_j|\psi\rangle\big|$.

a measurement is a projection: The projection operator allows us to determine both the probability of the measurement result and the state of the system after the measurement.

1.1 Projective measurement of a pure state (reminder)

The standard theory of projective measurement was conceptualized by von Neumann on the basis of the Born rule. We recall that, in quantum physics, an observable is described by a Hermitian operator $\hat{\mathcal{O}}$ and, for an isolated system, the state is described by a ket $|\psi\rangle$. The possible results of a measurement are the eigenvalues $\mathcal{O}_j$ of the operator $\hat{\mathcal{O}}$, associated with the eigenvectors $|\alpha_j^\nu\rangle$, assumed to be orthonormal. In these notations, j spans the set of distinct eigenvalues of $\hat{\mathcal{O}}$ and the other index ν accounts for possible degeneracies, see Fig. 1.

Single measurement

The result of an individual measurement is random. It yields $\mathcal{O}_j$ with probability

$$P_j = \big|\hat{\mathcal{P}}_j|\psi\rangle\big|^2 \quad \text{(Born's rule)} \quad , \tag{1}$$

where

$$\hat{\mathcal{P}}_j = \sum_\nu |\alpha_j^\nu\rangle\langle\alpha_j^\nu| \tag{2}$$

is the projection operator onto the eigenspace of $\hat{\mathcal{O}}$ associated with the eigenvalue $\mathcal{O}_j$. After a measurement giving the value $\mathcal{O}_j$, the state of the system is projected onto the corresponding eigenspace and renormalized,

$$|\psi\rangle \to |\psi'_{|j}\rangle = \frac{\hat{\mathcal{P}}_j|\psi\rangle}{\big|\hat{\mathcal{P}}_j|\psi\rangle\big|} \quad \text{(collapse principle)} \ . \tag{3}$$

This is the state of the system conditioned to the result of the measurement.

The average value of measurements performed on a set of systems all prepared in the same initial state is

$$\langle\mathcal{O}\rangle = \sum_j P_j\mathcal{O}_j = \langle\psi|\hat{\mathcal{O}}|\psi\rangle \ , \tag{4}$$

where we have used the spectral expansion $\hat{\mathcal{O}} = \sum_j \mathcal{O}_j\hat{\mathcal{P}}_j$ and the fact that a projector is a Hermitian operator.

Remarks:

1. The projection operators $\hat{\mathcal{P}}_j$ are sufficient to determine both the probability of a measurement result, Eq. (1), and the after-measurement state of the system conditioned to the result of the measurement, Eq. (3). We may thus say that the result of a measurement is a projection operator, hence the name *projection-valued measurement* (PVM).

2. A PVM provides classical information in the sense that it determines the measurement result of a particular observable $\mathcal{O}$. In particular, it disregards the coherences between the different subspaces $\mathcal{E}_j(\mathcal{O})$ associated to the different measurement results.

3. The collapse principle and the idempotent property of projectors, $\hat{\mathcal{P}}_j^2 = \hat{\mathcal{P}}_j$, ensure that while the result of a first measurement is random, any subsequent measurement performed immediately after the first one on the same system returns the same result.

4. A single measurement yields information about the system but it is important to understand which. If the measurement yields $\mathcal{O}_j$, we know that

   (i) The initial state had a nonvanishing component in $\mathcal{E}_j(\mathcal{O})$. This is a very weak but nonnegligible amount of information (see below).

   (ii) The final state is in $\mathcal{E}_j(\mathcal{O})$. If the value $\mathcal{O}_j$ is nondegenerate, the final state is exactly known (up to an irrevelant phase). Note that this offers a way to prepare the system in a given state, although it requires a post selection conditioned to the result of the measurement.

Unread measurement

So far, we have assumed that the measurement was read. Let us now assume that a measurement has been performed but that the result is not read by the observer. Although this seems like an academic question in the context of a measurement, we shall see later that unread measurements have very important applications for the interpretation of the dynamics of open systems and decoherence. In the case of an unread measurement, the state is projected in one of the other of the states $\hat{\mathcal{P}}_j|\psi\rangle/\big|\hat{\mathcal{P}}_j|\psi\rangle\big|$ with probability $P_j = |\hat{\mathcal{P}}_j|\psi\rangle|^2$. Considering the (classical) information we have, the after-measurement state is described by the density matrix

$$\hat{\rho}\,' = \sum_j P_j\,|\psi_{|j}\rangle\langle\psi_{|j}| = \sum_j P_j\,\frac{\hat{\mathcal{P}}_j|\psi\rangle\langle\psi|\hat{\mathcal{P}}_j}{\big|\hat{\mathcal{P}}_j|\psi\rangle\big|^2}\ , \tag{5}$$

that is

$$\hat{\rho}\,' = \sum_j \hat{\mathcal{P}}_j|\psi\rangle\langle\psi|\hat{\mathcal{P}}_j\ . \tag{6}$$

It can be checked that the after-measurement density matrix $\rho\,'$ fulfills all the necessary conditions for being a density matrix, see exercise 1b, page 21. The fact that a measurement turns the description of the system state from a ket (pure state) to a density matrix (mixed state in general) illustrates the acquisition of classical information.

1.2 Projective measurement of a mixed state

Assume now that even before the measurement we only have a partial knowledge of the system state. It is thus not defined by a ket any longer but by a density matrix. The latter may be written as

$$\hat{\rho} = \sum_n \Pi_n\,|\psi_n\rangle\langle\psi_n|\ , \tag{7}$$

where the $|\psi_n\rangle$'s form the family of possible states of the system, each with probability Π_n. We recall that the $|\psi_n\rangle$ are normalized but not necessarily mutually orthogonal.

Measurement probability

Let us now perform a measurement of the observable $\mathcal{O}$. As before, one can obtain either of the values $\mathcal{O}_j$ with probability $P_j = \sum_n \Pi_n \times P_{j|n}$, which may be written as

$$\boxed{P_j = \mathrm{Tr}\big(\hat{\rho}\,\hat{\mathcal{P}}_j\big)}\ , \tag{8}$$

The derivation is straightforward and consistent with the Born rule (1) in the case of a pure state, see exercise 2, page 21

After-measurement state and Bayesian inference

Let us now determine the state of the system after the measurement. Assume that we have obtained the value $\mathcal{O}_j$. Taking into account all the possible states of the system before the measurement, a naive calculation would yield

$$\tilde{\rho}'_{|j} = \sum_n \Pi_n \, \frac{\hat{\mathcal{P}}_j|\psi_n\rangle\langle\psi_n|\hat{\mathcal{P}}_j}{\big|\hat{\mathcal{P}}_j|\psi_n\rangle\big|^2} \; . \tag{9}$$

However, such a result raises inconsistencies. In particular, if a possible state $|\psi_n\rangle$ is orthogonal to the $\mathcal{E}_j(\hat{\mathcal{O}})$, we have $\hat{\mathcal{P}}_j|\psi_n\rangle = 0$ and the quantity $\hat{\mathcal{P}}_j|\psi_n\rangle\langle\psi_n|\hat{\mathcal{P}}_j/|\hat{\mathcal{P}}_j|\psi_n\rangle|^2$ is hill defined.

In fact, the above calculation ignores an important aspect of the measurement, namely that it provides information about the state of the system, which we did not necessarily have before the measurement. In particular, if we obtained the value $\mathcal{O}_j$, we acquire the information that the system was not in a state orthogonal to $\mathcal{E}_j(\hat{\mathcal{O}})$. This removes the hill-defined terms in $\tilde{\rho}'_{|j}$, but one can also do a much better use of the information gained by the measurement. Indeed, we may substitute the initial probability of the state $|n\rangle$ to its counterpart conditional to the measurement result, i.e.

$$\Pi_n \to P_{n|j} \; . \tag{10}$$

This approach, well known in classical probability theory, is called *Bayesian inference*. Although the quantity $P_{n|j}$ may seem difficult to calculate at first sight, it is easily found from the previously calculated $P_{j|n}$ using the *Bayes theorem*,

$$P_{a|b} = \frac{P_{b|a} P_a}{P_b} \; , \tag{11}$$

which is a direct consequence of the well-known relation $P_{a,b} = P_{a|b} \times P_b$. Using the relation $P_{j|n} = |\hat{\mathcal{P}}_j|\psi_n\rangle|^2$ and, Eq. (8), and the substitution (10), we then find

$$\hat{\rho}'_{|j} = \sum_n \underbrace{\frac{P_{j|n} \times \Pi_n}{P_j}}_{P_{n|j}} \, \frac{\hat{\mathcal{P}}_j|\psi_n\rangle\langle\psi_n|\hat{\mathcal{P}}_j}{\big|\hat{\mathcal{P}}_j|\psi_n\rangle\big|^2} = \frac{\sum_n \Pi_n \, \hat{\mathcal{P}}_j|\psi_n\rangle\langle\psi_n|\hat{\mathcal{P}}_j}{\mathrm{Tr}\big(\hat{\rho}\,\hat{\mathcal{P}}_j\big)} \; , \tag{12}$$

that is

$$\boxed{\hat{\rho}'_{|j} = \frac{\hat{\mathcal{P}}_j \,\hat{\rho}\, \hat{\mathcal{P}}_j}{\mathrm{Tr}\big(\hat{\rho}\,\hat{\mathcal{P}}_j\big)}} \; . \tag{13}$$

Moreover, if the measurement is not read, we have to average the conditional results $\hat{\rho}'_{|j}$ over the measurement results and we find $\hat{\rho}' = \sum_j P_j \times \hat{\rho}'_{|j}$, which yields

$$\boxed{\hat{\rho}' = \sum_j \hat{\mathcal{P}}_j \,\hat{\rho}\, \hat{\mathcal{P}}_j} \; . \tag{14}$$

Note that both $\hat{\rho}'_{|j}$ and $\hat{\rho}'$ fulfill all the properties of density matrices.

Quantum information approach

It is instructive to reconsider the above problem from the quantum information viewpoint. Assume that the system of interest (A) is coupled to another system (B). If the bipartite system ($A \otimes B$) is isolated the state may be written, without loss of generality,

$$|\Psi\rangle_{A\otimes B} \;=\; \sum_{1\leq n\leq \dim(\mathscr{H}_B)} c_n \;|\psi_n\rangle_A \otimes |\chi_n\rangle_B \;, \tag{15}$$

where the $|\psi_n\rangle_A$'s form a family of normalized states of $\mathcal{E}_A$ (not necessarily orthogonal) and the $|\chi_n\rangle_B$'s form an orthonormal basis of $\mathcal{E}_B$. Normalization of the bipartite state $|\Psi\rangle_{A\otimes B}$ implies

$$\sum_{1\leq n\leq \dim(\mathscr{H}_B)} |c_n|^2 = 1 \;. \tag{16}$$

The associated reduced density matrix of the sub-system A then reads as

$$\hat{\rho}_A = \sum_n \;|c_n|^2 \;\; |\psi_n\rangle\langle\psi_n| \;\; . \tag{17}$$

Let us now perform a measurement on the subsystem A only. Such a measure corresponds to an observable of the form $\hat{\mathcal{O}}_A \otimes \hat{\mathbb{1}}_B$. Similarly, the states of B being irrelevant, the projectors on the eigenspaces of the observable are written as $\hat{\mathcal{P}}_{j,A} \otimes \hat{\mathbb{1}}_B$. The measurement probabilities as well as the states obtained after measurement can then be obtained from the bipartite state (15). The probability of obtaining $\mathcal{O}_j$ is [see Eq. (1)]

$$P_j \;=\; \left|\big(\hat{\mathcal{P}}_{j,A} \otimes \hat{\mathbb{1}}_B\big)|\Psi\rangle\right|^2 \;=\; \mathrm{Tr}\big(\hat{\mathcal{P}}_j \hat{\rho}_A\big) \;, \tag{18}$$

see exercise 3, page 21. Equation (18) is nothing but Eq. (8).

Let us now asume that we have obtained the result $\mathcal{O}_j$. The bipartite state conditioned on this measurement result is [see Eq. (3)]

$$|\Psi'_{|j}\rangle = \frac{\big(\hat{\mathcal{P}}_{j,A} \otimes \hat{\mathbb{1}}_B\big) \sum_n c_n \;|\psi_n\rangle_A \otimes |\chi_n\rangle_B}{\sqrt{P_j}} = \frac{\sum_n c_n \;\big(\hat{\mathcal{P}}_j |\psi_n\rangle\big)_A \otimes |\chi_n\rangle_B}{\sqrt{P_j}} \;. \tag{19}$$

It corresponds to the reduced density matrix

$$\hat{\rho}'_{A|j} = \frac{\sum_n |c_n|^2 \hat{\mathcal{P}}_j \,|\psi_n\rangle\langle\psi_n|\, \hat{\mathcal{P}}_j}{P_j} \;, \tag{20}$$

which is nothing but Eq. (13).

We thus find by the quantum information approach the same results as obtained by the density matrix approach. It is quite remarkable that, here, we did not need to explicitely use Bayesian inference, i.e. we did not need to modify by hand the probabilities of the various possible states of A. The reason is that the projection is performed on the bipartite space $\mathcal{E}_A \otimes \mathcal{E}_B$ and it is the ket in this space that is globally renormalized. In the subspace of $\mathcal{E}_A$, this corresponds to projecting the reduced density matrix, $\hat{\rho} \to \hat{\mathcal{P}}_j \hat{\rho} \hat{\mathcal{P}}_j$, and then to renormalize by $\mathrm{Tr}\big(\hat{\mathcal{P}}_j \hat{\rho} \hat{\mathcal{P}}_j\big)$, that is to say Eq. (13).

2 Quantum nondemolition measurements

We now study a class of measurements specific to quantum mechanics, called *quantum nondemolition measurements* (QND measurements).

2.1 Motivation

Usual measurements in quantum physics are *destructive*. This contains two realities that should not be mixed up:

On the one hand, by virtue of the Born collapse rule, the state of the system is modified by the measurement. There is strictly nothing we can do against it. To the best of our knowledge, the collapse rule is a fundamental aspect of quantum physics and, up to now, no experience has ever challenged it. It is worth noticing that this is a characteristic of quantum physics. In classical physics, the coupling to a measurement device may of course affect the system's dynamics but it is a perturbation, which can, in principle, be arbitrarily reduced. I contrast, a PVM in quantum physics, dramatically affects the system state, whatever the coupling strength to the measurement device.

On the other hand, the physical integrity of the system may be affected by the measurement process. In extreme cases, it does not even exist any longer after the measurement. The system is thus *demolished*. In fact, standard quantum measurements are demolition measurements. It includes the Stern and Gerlach experiment, the Young slit experiment with photons or electrons, photodetection of single or multi-photon states, as well as time-of-flight imaging in ultracold atoms, to name a few. However, in contrast to wavefunction collapse, this is a practical rather than fundamental issue and, one can in principle overcome it. In the following, we shall describe a general approach to realize a full QND measurement using quantum entanglement.

Before proceeding, let us stress that standard (demolition) measurements have many disadvantages: Firstly, as mentionned above, an important property of PVMs in quantum physics is that, although the result of a measurement is random, two subsequent measurements of the same quantity always yield the same result. This is ensured by the idempotent property of projectors ($\hat{\mathcal{P}}_j^2 = \hat{\mathcal{P}}_j$). However, this pivotal property for the consistency of quantum physics cannot be tested experimentally if the system is destroyed after the first measurement. Secondly, as already mentioned above, an efficient method to prepare a system in a given state is to measure it in this state: Assume you have a qubit in the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ and measure it in the basis $\{|0\rangle, |1\rangle\}$; If the measurement returns 0, the qubit is now for sure in $|0\rangle$; If the measurement returns 1, the qubit is for sure in $|1\rangle$. However, if the system is destroyed by the measurement, it is of little use for subsequent quantum information processes! Finally, it is not possible to track the time evolution of a quantum system. This is most often not a serious issue as a measurement, wathever it is, would alter the unitary evolution of the system. In contrast, it is a serious issue to demonstrate effects induced by repeted measurements such as the Zeno effect for instance.

2.2 QND measurement: Formal definition

Let us first give a formal definition. Although it will be of little use to us in the remainder of the discussion and, *in fine*, does not add much to measurement theory, it is the definition that is given in many monographs:

A QND measurement is a measurement that can be repeated a large number of times, at arbitrary times, and always returns the same result.

This definition has two immediate consequences. On the one hand, the system must be available after the measurement. On the other hand, it requires that the evolution of the system between two measurements maintains it in the eigenspace corresponding to the result of the first measurement, as illustrated on Fig 2.

Property
A quantum measure is QND if and only if

(i) The system is available after the measurement;

(ii) The observable $\hat{\mathcal{O}}$ commutes with the evolution operator $\hat{U}(t)$ at any time, or equivalently, with the Hamiltonian $\hat{H}(t)$ at any time.

Note that, the second requirement may be smoothed if we only require that the measurements are repeated at the discrete times t_1, t_2, t_3, ... Then, it is sufficient to show that $\hat{U}(t_n)$ commutes with $\hat{\mathcal{O}}$ for any j.

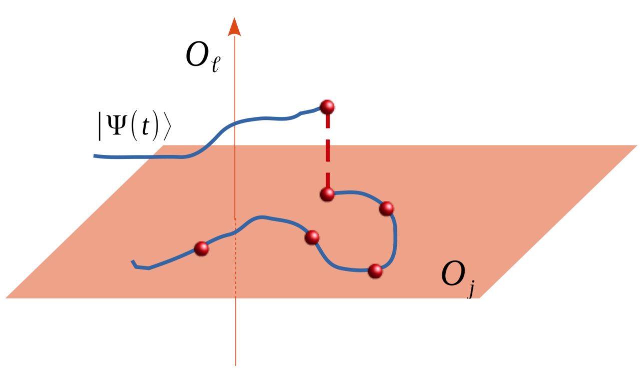

Figure 2: Sketch of a QND measurement, according to the definition of Sec. 2.2. The first measurement projects the state on the eigenspace of the observable corresponding to the obtained result. The subsequent evolution – unitary evolution (solid blue line) interrupted by measurements (red points) – maintains the state in the same eigenspace.

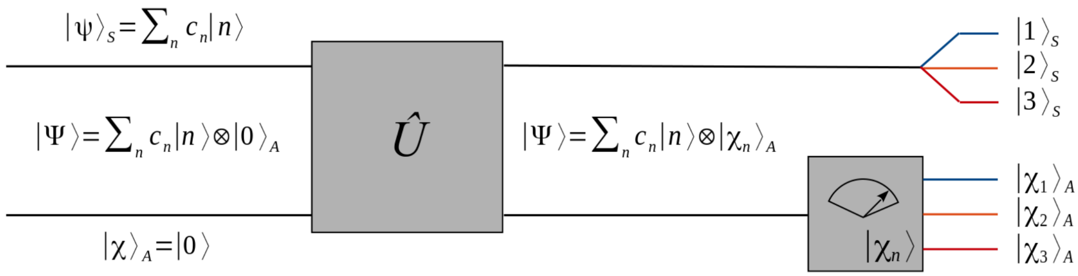

Figure 3: Principle of a QND measurement (see text).

Examples of QND or non-QND measurements are listed below:

- For a free particle, governed by the Hamiltonian $\hat{H} = \hat{\boldsymbol{p}}^2/2m$, $\hat{\boldsymbol{p}}$ is a QND measurement but $\hat{\boldsymbol{r}}$ is not.
- For a harmonic oscillator, $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$, neither $\hat{x}$ or $\hat{p}$ is QND. In turn, the excitation number $\hat{n} = \hat{a}^\dagger\hat{a}$ is QND.
- For a 1/2-spin in a magnetic field $\boldsymbol{B} = B\boldsymbol{e}_x$, $\hat{S}_x$ is QND but $\hat{S}_y$ and $\hat{S}_z$ are not.
- For any system, the energy is QND.

In all these examples, it is assumed that the system is available after the measurement.

2.3 QND measurement in practice

From now on, we adopt a more ”relaxed” definition of QND measurements, consisting in requiring only the condition (i) to hold, namely that the system is available after the measurement. The idea to perform a QND measurement is then to couple the system of interest to an auxilliary system and perform a standard, i.e. demolishing, measurement of this auxilliary system, without ”touching” the system of interest. Of course, the aim is to carry out a true measurement of the system state, i.e. to know in which state it is (after the measurement).

General principle

The principle of a QND measurement is shown as a quantum circuit on Fig. 3, and may be summarized as follows: Assume that the system S you want to measure is in the general superposition state $|\psi\rangle = \sum_n c_n |n\rangle$, where $\{|n\rangle\}$ is an orthonormal basis. Then prepare an ancilla A into some reference state $|0\rangle$ and couple it to the system using an appropriate evolution operator $\hat{U}$, so as to create an entangled state of the form

$$|\Psi\rangle = \sum_n c_n |n\rangle \otimes |\chi_n\rangle , \tag{21}$$

where $\{|\chi_n\rangle\}$ forms an orthonormal family. This requires that the dimension of the ancilla Hilbert space is at least equal to that of the system of interest. Then, assume you can perform a standard PVM in the $\{|\chi_n\rangle\}$ familly by coupling the ancilla to a measurement apparatus, without touching the system. The probability to find $|\chi_n\rangle$ is $P_n = |\hat{\mathcal{P}}_n|\Psi\rangle|^2 = |c_n|^2$, where $\hat{\mathcal{P}}_n = \hat{\mathbb{1}}_S \otimes \hat{\mathcal{P}}_{A,n}$ is the projector onto the ancilla Hilbert space associated to $|\chi_n\rangle$. This probability is equal to that of the system being initially in the state $|n\rangle$. Moreover, assuming you find $|\chi_n\rangle$, the bipartite state is projected onto the conditional pure state

$$|\Psi\rangle \to |\Psi'_{|n}\rangle = \frac{\hat{\mathcal{P}}_n|\Psi\rangle}{\left|\hat{\mathcal{P}}_n|\Psi\rangle\right|} = |n\rangle \otimes |\chi_n\rangle \ . \tag{22}$$

After the measurement, the system and the ancilla are no longer entangled and the system is in the pure state $|n\rangle$. More precisely, in many cases, the state $|n\rangle_S \otimes |\chi_n\rangle_A$ is actually improper because the ancilla is physically destroyed by the PVM, and we should rather restrict the after-measurement state to that of the system of interest, i.e. $|n\rangle$. Note that neither the coupling $\hat{U}$ nor the PVM on A has destroyed S, which is still available for subsequent processes. In the end, we have thus realized an effective PVM on S without destroying it, i.e. a QND measurement of S.

The feasibility of such a QND measurement requires that we are able to realize an appropriate gate $\hat{U}$. We can formally convince ourselves that this is indeed possible by noting that the operation

$$\sum_n c_n |n\rangle \otimes |0\rangle \longrightarrow \sum_n c_n |n\rangle \otimes |\chi_n\rangle \tag{23}$$

can be prolongated into a unitary operation. Indeed, the states $|n\rangle$ form an orthonormal basis of $\mathcal{E}_S$, so that the both $\{|n\rangle \otimes |0\rangle , n \in \mathbb{N}\}$ and $\{|n\rangle \otimes |\chi_n\rangle , n \in \mathbb{N}\}$ form orthonormal families of $\mathcal{E}_{S\otimes A}$, which can thus be completed into orthonormal bases. Note that this would also hold if the $|\chi_n\rangle$'s were normalized but not orthogonal. Here, we have nevertheless assumed that they are orthonormal, so that there can exist an observable to be measured of which they are eigenstates. Note also that it is not necessary for the states of the system S and the ancilla A be the same. It is enough to have the reading code $|n\rangle_S \leftrightarrow |\chi_n\rangle_A$. It is not even necessary that the two systems are of the same nature.

Example

Let us turn to a concrete realization. Assume that the system of interest S is a qubit. To perform a QND measurement, we need an ancilla that lives in a Hilbert space at

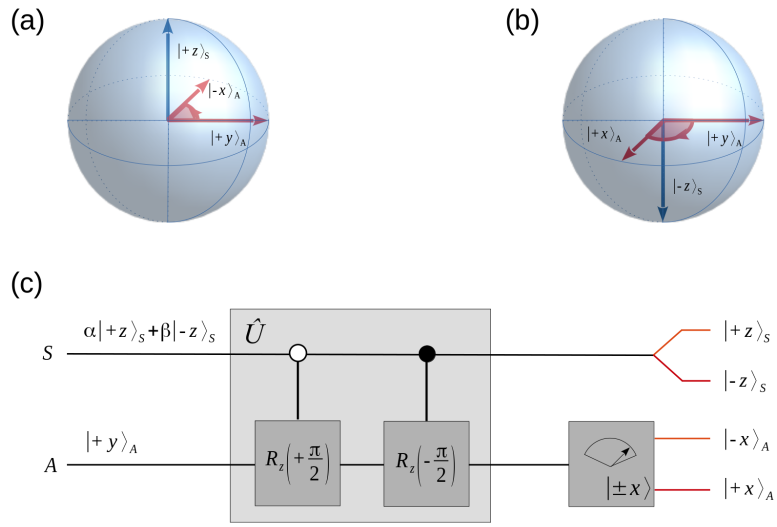

Figure 4: (a) and (b) Effect of an antiferromagnetic coupling between two spins. It shows a $\pi/2$ pulse on the ancilla spin A (red arrows), initially in the state $|+y\rangle_A$, depending on whether the spin $\mathcal{S}$ (blue arrow) is (a) in the state $|+z\rangle_{\mathcal{S}}$ or (b) in the state $|-z\rangle_{\mathcal{S}}$. (c) Quantum circuit realizing the QND measurement of the spin state of S, using an ancilla spin A.

least as large, so at least another qubit. For simplicity, we shall use here a 1/2-spin representation. Assume that the system of interest S is in any state

$$|\psi\rangle_S = \alpha\,|+z\rangle_S + \beta\,|-z\rangle_S \ . \tag{24}$$

The ancilla is then prepared in the state

$$|+y\rangle_A = \frac{|+z\rangle_A + i\,|-z\rangle_A}{\sqrt{2}} \tag{25}$$

and we apply the Ising-type coupling Hamiltonian

$$\hat{H} = J\hat{S}^z_S \cdot \hat{S}^z_A = \frac{\hbar g}{2}\hat{\sigma}^z_S \cdot \hat{\sigma}^z_A \ . \tag{26}$$

For the sake of concreteness, the coupling is assumed to be antiferromagnetic ($J > 0$) so that $g = \hbar J/2 > 0$, but similar results are obtained with negative g. Since $\hat{H}$ is diagonal in the computional basis $\{|\pm z\rangle_S \otimes |\pm z\rangle_A\}$, its action is trivial:

- If S is in the state $|+z\rangle$, $\hat{H}$ acts on A as a magnetic field along the z axis. Since $g > 0$, it induces a counter-clockwise precession of the qubit A around z, see Fig. 4(a). The interaction strength and times are tuned so as to realize a $\pi/2$ pulse, i.e. $t = \pi/2g$. It realizes the transformation[1]

$$|+y\rangle_A \longrightarrow \frac{1}{\sqrt{i}}\,|-x\rangle_A \ . \tag{27}$$

[1] The phase factor is found by integrating the Schrödinger equation. It is, however, unimportant here since it does not alter the QND measurement.

- If S is in the state $|-z\rangle$ the effective magnetic field acting on the ancilla A is inverted and the rotation on the Bloch sphere is now clockwise, see Fig. 4(b). It realizes the transformation[1]

$$|+y\rangle_A \longrightarrow \sqrt{i}\,|+x\rangle_A \ . \tag{28}$$

This process may be represented by the quantum circuit shown on Fig. 4(c). Since the coupling is unitary, at the end of the interaction process, the bipartite system $S \otimes A$ ends up in the entangled state

$$|\Psi_{S\otimes A}\rangle = \frac{\alpha}{\sqrt{i}}\,|+z\rangle_S \otimes |-x\rangle_A + \sqrt{i}\beta\,|-z\rangle_S \otimes |+x\rangle_A \,. \tag{29}$$

We can now perform a PVM of the spin of A along the x axis:

- We find the ancilla in $|-x\rangle_A$ with probability $|\alpha|^2$ and the system S is then projected onto $|+z\rangle_S$;
- We find the ancilla in $|+x\rangle_A$ with probability $|\beta|^2$ and the system S is then projected onto $|-z\rangle_S$.

We have hence realized a QND meaurement of the spin state of S.

2.4 QND measurement in cavity quantum electrodynamics

We now discuss a concrete implementation of a QND measurement in a cavity quantum electrodynamics (CQED) experiment. It essentially realizes the scheme proposed above. The idea is to measure the photon number in a cavity C, without destroying the photons, via Rydberg atoms. The system of interest is the photon field in the cavity mode and the ancilla is a Rydberg atom. Here we focus on a simplified discussion, which contains the main ingredients of the experiment. In the following, we assume that the cavity mode has angular frequency ω and contains at most one photon. The most general state reads as

$$|\psi\rangle_C = \alpha\,|0\rangle + \beta\,|1\rangle \ . \tag{30}$$

The Rydberg atoms are considered as two-level systems, with ground state $|\mathrm{g}\rangle$, excited state $|\mathrm{e}\rangle$, and atomic angular frequency ω_A. The experimental apparatus is shown on Fig. 5.

The Rydberg atoms are emitted from the box B. They are all prepared in the excited state $|\mathrm{e}\rangle$ using an excitation cascade, and injected one by one in the apparatus. Each atom is then turned into a 50%-50% superposition of the ground and excited states. This is realized by a $\pi/2$ pulse in the cavity R_1. The latter contains a resonant quasi-classical field, at the angular frequency $\omega_1 = \omega_A$. The atom undergoes a Rabi oscillation of unit

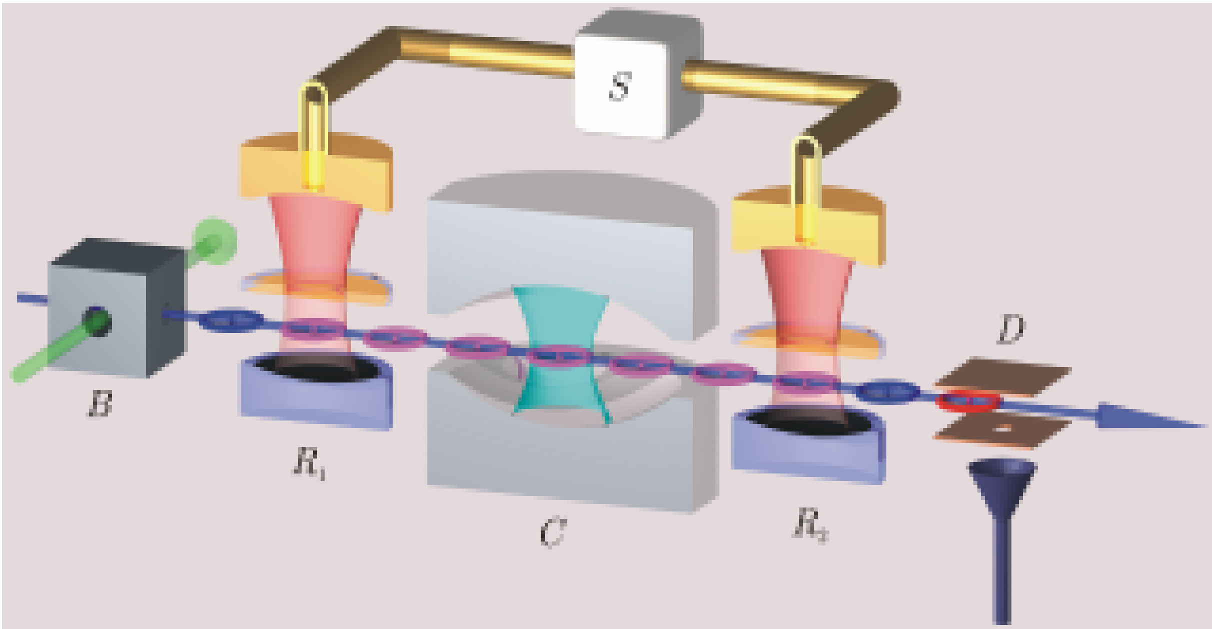

Figure 5: Experimental device for the quantum nondemolition measurement of photons in a cavity.

amplitude and period $2\pi/\Omega_R$ where Ω_R is the Rabi angular frequency. By setting the interaction time to $T_R = \pi/2\Omega_R$, the atom ends up into the state

$$|\chi'\rangle_A = \frac{-i\,|\mathrm{g}\rangle + \mathrm{e}^{-i\omega_A T_R}\,|\mathrm{e}\rangle}{\sqrt{2}} \ . \tag{31}$$

To simplify the calculations, we assume here that $\omega_A T_R \equiv 0\ [2\pi]$, so that

$$|\chi'\rangle_A = -i\,\frac{|\mathrm{g}\rangle + i\,|\mathrm{e}\rangle}{\sqrt{2}} \ . \tag{32}$$

This transformation corresponds to a rotation of the Bloch vector around the x axis from $|\mathrm{e}\rangle = |-z\rangle$ to $-i\,|+y\rangle$, in spin notations.

The atom then enters the cavity C and interacts with the cavity field to be measured. We assume it reads as in Eq. (30) when the atom enters the cavity C. The cavity mode has angular frequency ω, detuned from the atomic resonance by $\delta = \omega - \omega_A < 0$. The coupling Hamiltonian reads as $\hat{H} = \hat{H}_A + \hat{H}_C + \hat{H}_{\mathrm{int}}$, where $\hat{H}_A = \hbar\omega_A\,|\mathrm{e}\rangle\langle\mathrm{e}|$ is the Hamiltonian of the atom, $\hat{H}_C = \hbar\omega\hat{a}^\dagger\hat{a}$ that of the cavity mode, with $\hat{a}$ the photon annihilation operator, and $\hat{H}_{\mathrm{int}} = \frac{\hbar\Omega_0}{2}\left(|\mathrm{e}\rangle\langle\mathrm{g}|\,\hat{a} + |\mathrm{g}\rangle\langle\mathrm{e}|\,\hat{a}^\dagger\right)$ is the coupling term. In the weak coupling regime considered here, $|\delta| \gg \Omega_n$, the Hamiltonian can be solved using perturbation theory. To lowest order, the eigenstates, $|\mathrm{g},0\rangle$, $|\mathrm{g},n+1\rangle$, and $|\mathrm{e},n\rangle$ for $n \in \mathbb{N}$, are unperturbed. The ground state $|\mathrm{g},0\rangle$ is isolated and has energy $E_{\mathrm{g},0} = 0$. The energies of all other states are calculated using second-order perturbation theory, which yields

$$E_{\mathrm{g},n+1} \simeq E^0_{\mathrm{g},n+1} - \frac{\hbar\Omega_0^2}{4|\delta|}(n+1) \qquad \text{and} \qquad E_{\mathrm{e},n} \simeq E^0_{\mathrm{e},n} + \frac{\hbar\Omega_0^2}{4|\delta|}(n+1)\ , \tag{33}$$

with $E^0_{\mathrm{g},n+1} = \hbar\omega(n+1)$ and $E^0_{\mathrm{e},n} = \hbar\omega_A + \hbar\omega n$ the unperturbed energies. Now, the state of the atom-field system at the input of cavity C reads as

$$\begin{aligned} |\Psi'\rangle_{A\otimes C} &= \left(\frac{-i\,|\mathrm{g}\rangle + |\mathrm{e}\rangle}{\sqrt{2}}\right) \otimes \big(\alpha\,|0\rangle + \beta\,|1\rangle\big) \qquad (34)\\ &= \frac{\alpha}{\sqrt{2}}\big(-i\,|\mathrm{g},0\rangle + |\mathrm{e},0\rangle\big) + \frac{\beta}{\sqrt{2}}\big(-i\,|\mathrm{g},1\rangle + |\mathrm{e},1\rangle\big)\ . \end{aligned}$$

After an interaction time T_C, we thus obtain

$$\begin{aligned} |\Psi''\rangle_{A\otimes C} &= \frac{\alpha}{\sqrt{2}}\big(-i\,|\mathrm{g},0\rangle + \mathrm{e}^{-i\omega_A T_C}\mathrm{e}^{-i\phi_0}\,|\mathrm{e},0\rangle\big) \qquad (35)\\ &\quad + \frac{\beta}{\sqrt{2}}\mathrm{e}^{-i\omega T_C}\big(-i\mathrm{e}^{+i\phi_0}\,|\mathrm{g},1\rangle + \mathrm{e}^{-i\omega_A T_C}\mathrm{e}^{-i2\phi_0}\,|\mathrm{e},1\rangle\big)\ , \end{aligned}$$

with $\phi_0 = \Omega_0^2 T_C/4|\delta|$. By setting the system so that $\phi_0 = \pi/2$ and assuming $\omega_A T_C \equiv 0\ [2\pi]$ as before to simplify the calculations, we finally obtain

$$|\Psi''\rangle_{A\otimes C} = -i\alpha\frac{|\mathrm{g}\rangle + |\mathrm{e}\rangle}{\sqrt{2}} \otimes |0\rangle + \beta\mathrm{e}^{-i\omega T_C}\frac{|\mathrm{g}\rangle - |\mathrm{e}\rangle}{\sqrt{2}} \otimes |1\rangle\ . \tag{36}$$

We have thus generated an atom-field entangled state. The zero and one photon states, $|0\rangle$ and $|1\rangle$, are coupled to the atomic states $(|\mathrm{g}\rangle + |\mathrm{e}\rangle)/\sqrt{2} = |{+x}\rangle$ and $(|\mathrm{g}\rangle - |\mathrm{e}\rangle)/\sqrt{2} = |{+x}\rangle$, respectively. By performing a standard PVM of the atomic state in this basis, we realize in principle a QND measurement of the photonic state. Note that the probability amplitudes have been preserved, up to a phase, by the coupling, so that we obtain $|0\rangle$ and $|1\rangle$ with the probabilities $|\alpha|^2$ and $|\beta|^2$ of the initial cavity state.

It is, however, not straightforward to measure the atom in the appropriate basis. We know how to measure in the $\{|\mathrm{g}\rangle\,, |\mathrm{e}\rangle\}$ basis. We may for instance subject the atom to a field resonant with a $|\mathrm{e}\rangle \to |2\rangle$ transition, but off resonant with any transition from $|\mathrm{g}\rangle$, and perform a π pulse. Should a photon be absorbed, the atom was in $|\mathrm{e}\rangle$; Otherwise it was in $|\mathrm{g}\rangle$. Such an approach cannot be extended to measure in $|{\pm x}\rangle$ because the two states have the same energy, at least without forbidding the transition from one of the states by some symmetries.

To circumvent this difficulty, we may rotate the atomic state, independently of the photonic state, before measuring it. This is realized by the cavity R_2. It contains a resonant quasi-classical field, similar to that in cavity R_1 but with a relative phase $-\pi/2$. It realizes a rotation of the atomic state by $\pi/2$ around the y axis in a counter-clockwise direction,

$$\frac{|\mathrm{g}\rangle + |\mathrm{e}\rangle}{\sqrt{2}} \longrightarrow +\,|\mathrm{g}\rangle \qquad \text{and} \qquad \frac{|\mathrm{g}\rangle - |\mathrm{e}\rangle}{\sqrt{2}} \longrightarrow -\,|\mathrm{e}\rangle\ , \tag{37}$$

and the atom-cavity state now reads as

$$|\Psi'''\rangle_{A\otimes C} = -i\alpha\,|\mathrm{g}\rangle \otimes |0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|\mathrm{e}\rangle \otimes |1\rangle \ . \tag{38}$$

We can now measure the atomic state in the standard basis $\{|\mathrm{g}\rangle, |\mathrm{e}\rangle\}$:

- The atom is measured in $|\mathrm{g}\rangle$ with probability $|\alpha|^2$, and the cavity field is then projected onto $|0\rangle$;
- The atom is measured in $|\mathrm{e}\rangle$ with probability $|\beta|^2$, and the cavity field is then projected onto $|1\rangle$.

A QND measurement of the photon number in the cavity mode has thus been realized.

This apparatus has been applied to a variety of QND measurements, including the determination photon numbers [2] and well as the observation of the birth and death of a thermal photon in the cavity [3].

3 Generalized measurements (POVM)

In fact, standard *à la von Neumann* PVMs are only a particular example of a more general class of measurements in quantum physics. Their framework is often too restrictive and not adapted to many real experimental situations met in the field of quantum information. We have actually met a concrete example above in the discussion of QND measurements in CQED: If we do not apply the last rotation of the atomic state on the Bloch sphere as realized by the cavity R_2, then the measurement of the atomic state $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$ projects the cavity state onto $-i\alpha\,|0\rangle \pm \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle$, respectively. In general, these states are not orthogonal and therefore cannot be associated to the orthogonal eigenspaces of a Hermitian observable $\hat{\mathcal{O}}$ acting in the Hilbert space of the cavity C.

There are many other examples. For instance, the detection of the state of an atom can be realized by ionization: The atom is subjected to an electric field gradient. The ionization starts at the point where the energy transfer equals the ionization energy. Since this ionization energy depends on the atomic state, we can deduce the latter from the position where the ionization started. The operators describing the action of the measurement conditioned to its result read as

$$\hat{M}_\mathrm{g} = |\mathrm{ion}\rangle\langle\mathrm{g}| \qquad \text{and} \qquad \hat{M}_\mathrm{e} = |\mathrm{ion}\rangle\langle\mathrm{e}| \ , \tag{39}$$

which are definitely not projectors. In particular, $M_\mathrm{g}^2 = M_\mathrm{e}^2 = 0$, and the measurement cannot be repeted, although the system stil exists after the meaurement.

Another example is a photon counter in a given mode of the radiation field. The absorption of a photon extracts an electron from the detector, which may then be amplified and measured. The resulting electric current is proportional to the number of absorbed photons and can therefore be determined. In all cases, the photons are all absorbed, a process described by the operators

$$\hat{M}_n = |0\rangle\langle n| \ . \tag{40}$$

Again, these operators are clearly not projectors in the Hilbert space of the radiation mode. In particular, $\hat{M}_n^2 = \delta_{n,0} |0\rangle\langle 0|$.

3.1 Describing a generalized measure from the unitary evolution of a larger system

In order to describe the most general quantum measurement processes, note that the measured system cannot be separated from the measuring apparatus. Then, consider a system of interest S measured by the measuring apparatus M, called the *meter*. The meter, which we do not explicitly write here, is either a standard macroscopic measuring device, or in the case of QND measurements, the same device coupled to the ancilla A.

The system S is prepared in any state $|\psi\rangle_S$ and the meter in the reference state $|0\rangle_M$,

$$|\Psi\rangle_{S\otimes M} = |\psi\rangle_S \otimes |0\rangle_M \ . \tag{41}$$

Then, let the system and the meter interact. It generally creates an entangled state, which can be written, without loss of generality, in the form

$$|\Psi'\rangle_{S\otimes M} = \hat{U} |\Psi\rangle_{S\otimes M} = \sum_m \left(\hat{M}_m |\psi\rangle_S\right) \otimes |m\rangle_M \ , \tag{42}$$

where the states $|m\rangle_M$ describe an orthonormal basis of the meter's Hilbert space and

$$\hat{M}_m = {}_M\langle m| \hat{U} |0\rangle_M \, . \tag{43}$$

Note that $\hat{M}_m$ is an operator on $\mathcal{E}_S$.

The states $|m\rangle_M$ represent the set of possible results of a measurement read on the meter while the so-called Kraus operators $\hat{M}_m$ describe the action of the measurement process on the system S conditioned to the measurement result read on M. In the case of a standard von Neumann measurement described by an observable $\hat{\mathcal{O}}$ on $\mathcal{E}_S$, these operators are nothing but the projectors $\hat{\mathcal{P}}_j$ on the eigensubspaces of $\hat{\mathcal{O}}$. Here, the operators $\hat{M}_m$ are in general not projectors, see the examples above. Note in particular that the number of such operators is equal to the dimension of the Hilbert space of the meter, $\dim(\mathcal{E}_M)$, possibly larger than the dimension of the Hilbert space of the system of interest, $\dim(\mathcal{E}_S)$.

The operators $\hat{M}_m$ are, however, not completely arbitrary. Indeed, the coupling between the system and the meter is described by a unitary evolution operator $\hat{U}$, thus preserving the norm of the bipartite state $|\Psi\rangle_{S\otimes M}$. It follows the relation

$$1 = \langle\Psi'|\,|\Psi'\rangle = \sum_m {}_S\langle\psi|\,\hat{M}_m^\dagger \hat{M}_m\,|\psi\rangle_S\,,$$

since the states $|m\rangle_M$ form an orthonormal family. This relation being valid for any state of the system S, it implies the *completeness relation*

$$\boxed{\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{\mathbb{1}}_S}\,. \tag{44}$$

This identity generalizes the completeness relation on a complete set of orthogonal projectors, $\sum_j \hat{\mathcal{P}}_j = \sum_j \hat{\mathcal{P}}_j^2 = \hat{\mathbb{1}}_S$. More precisely, we shall see below that the Kraus operators $\hat{M}_m$ are the exact counterparts of the projection operators $\hat{\mathcal{P}}_j$ in generalized measurement theory.

We shall see that generalized quantum measurements are completely determined by the set of Kraus operators $\hat{M}_m$, just as the set of projectors $\hat{\mathcal{P}}_j$ determine PVMs. Generalized measurements are often called *positive operator valued measurements (POVM)*. Note, however, that the Kraus operators $\hat{M}_m$ need not be positive operators. Here the term "positive" refers to the quantities $\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$, which are indeed positive operators and are sufficient to determine the measurement probabilities, see below.

3.2 Generalized measurement of a pure state

Assume that the result of the measurement on the meter is read. This process is described by a PVM performed on M. Conditional to the measurement result, the state of the meter is projected onto one or the other of the states $|m\rangle$. According to the collapse principle applied to $S\otimes M$, it follows that the after-measurement state is given by the renormalized projection on $|m\rangle_M$, described by $\hat{\mathcal{P}}_m = \hat{\mathbb{1}}_S \otimes |m\rangle\langle m|_M$, i.e.

$$|\Psi'_{|m}\rangle = \frac{\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M}{\left|\hat{M}_m|\psi\rangle_S \otimes |m\rangle_M\right|}\,. \tag{45}$$

This state being a product state of S and M, the states of both subsystems are well defined and in particular

$$\boxed{|\psi'_{|m}\rangle_S = \frac{\hat{M}_m|\psi\rangle_S}{\left|\hat{M}_m|\psi\rangle_S\right|}}\,. \tag{46}$$

This is the state of S conditionned to the result of the measurement performed on M.

According to the Born rule, the probability of measuring m is

$$P_m = \left| \hat{M}_m |\psi\rangle_S \otimes |m\rangle_M \right|^2 , \tag{47}$$

that is

$$\boxed{P_m = \left| \hat{M}_m |\psi\rangle \right|^2} . \tag{48}$$

Note that it is crucial that the states $|m\rangle_M$ form an orthonormal set. This allows us to consider them as the results of a standard PVM on the meter, as well as for Eq. (48) to hold.

Now, if the result of the measurement is unread, the after-measurement state of S is described by the density matrix

$$\hat{\rho}\,' = \sum_m P_m \, |\psi'_{|m}\rangle\langle\psi'_{|m}| = \sum_m P_m \, \frac{\hat{M}_m \, |\psi\rangle\langle\psi| \, \hat{M}_m^\dagger}{\left| \hat{M}_m |\psi\rangle \right|^2} , \tag{49}$$

that is

$$\boxed{\hat{\rho}\,' = \sum_m \hat{M}_m \, |\psi\rangle\langle\psi| \, \hat{M}_m^\dagger} . \tag{50}$$

These results directly generalize those of Sec. 1.1 by replacing the projection operators $\hat{\mathcal{P}}_j$ by the Kraus operators $\hat{M}_m$ or $\hat{M}_m^\dagger$ depending on whether they are associated to a ket or a bra. Note that the same result can be obtained by computing the partial trace on the meter states of the bipartite density matrix after interaction, $|\Psi'\rangle\langle\Psi'|$.

3.3 Generalized measurement of a mixed state

Let us now generalize the notion of POVMs to the case where the system of interest S is in a mixed state, described by the density matrix

$$\hat{\rho} = \sum_n \Pi_n \, |\psi_n\rangle\langle\psi_n| \ . \tag{51}$$

To simplify the discussion, we shall here use a *purification approach*. It consists in describing a mixed state of S as a ket in an extended space. It is easier than the direct density-matrix approach because the Bayesian inference is built in, as discussed at the end of Sec. 1.2. Concretely, assume that the system S is in an entangled state with some environment E,

$$|\Psi\rangle_{S\otimes E} = \sum_n c_n \, |\psi_n\rangle_S \otimes |\chi_n\rangle_E \ , \tag{52}$$

where the coefficients c_n are such that $|c_n|^2 = \Pi_n$, with any phase, and the $|\chi_n\rangle$'s form an orthonormal family of $\mathcal{E}_E$. We know that we can always do this as long as we only consider measurements performed on S and not on E since no measurement on S can distinguish a mixed state from a state that is entangled with an unobserved system. The tripartite state before the measurement then reads as

$$|\Psi\rangle_{S\otimes E\otimes M} = \sum_n c_n \, |\psi_n\rangle_S \otimes |\chi_n\rangle_E \otimes |0\rangle_M \ . \tag{53}$$

In practice, E can be a real or fictitious environment of the system. It does not matter for what follows.

We then perform a measurement on S alone by coupling it to the measuring device. The tripratite state resulting from the coupling reads as

$$|\Psi'\rangle_{S\otimes E\otimes M} = \sum_m \left[\sum_n c_n \left(\hat{M}_m \, |\psi_n\rangle_S\right) \otimes |\chi_n\rangle_E\right] \otimes |m\rangle_M \ . \tag{54}$$

Note that we assume here that the measuring device only acts on S, so the interaction does not affect the environment E. Nevertheless, the entanglement created between S and M induces an entanglement between E and M. The state of the system $S \otimes E$ conditioned to the result of the measurement performed on M reads as

$$|\Psi'_{|m}\rangle_{S\otimes E} = \frac{\sum_n c_n \left(\hat{M}_m \, |\psi_n\rangle_S\right) \otimes |\chi_n\rangle_E}{\left|\sum_n c_n \left(\hat{M}_m \, |\psi_n\rangle_S\right) \otimes |\chi_n\rangle_E\right|} \ . \tag{55}$$

Tracing over the environment degrees of freedom, we then obtain the density matrix of S

$$\hat{\rho}'_{|m} = \frac{\sum_n |c_n|^2 \hat{M}_m \, |\psi_n\rangle\langle\psi_n| \, \hat{M}_m^\dagger}{\sum_n |c_n|^2 \, \langle\psi_n| \, \hat{M}_m^\dagger \hat{M}_m \, |\psi_n\rangle} \ , \tag{56}$$

that is

$$\boxed{\hat{\rho}'_{|m} = \frac{\hat{M}_m \, \hat{\rho} \, \hat{M}_m^\dagger}{\mathrm{Tr}\left(\hat{M}_m \, \hat{\rho} \, \hat{M}_m^\dagger\right)}} \ . \tag{57}$$

The probability of measuring m is

$$P_m = \left|\sum_n c_n \left(\hat{M}_m \, |\psi_n\rangle_S\right) \otimes |\chi_n\rangle_E\right|^2 = \sum_n |c_n|^2 {}_S\langle\psi_n| \, \hat{M}_m^\dagger \hat{M}_m \, |\psi_n\rangle = \mathrm{Tr}\left(\hat{\rho} \, \hat{M}_m^\dagger \, \hat{M}_m\right) \ . \tag{58}$$

It yields

$$\boxed{P_m = \mathrm{Tr}\left(\hat{M}_m \, \hat{\rho} \, \hat{M}_m^\dagger\right) = \mathrm{Tr}\left(\hat{E}_m \, \hat{\rho}\right)} \ , \tag{59}$$

where $\hat{E}_m = \hat{M}_m^\dagger \hat{M}_m$.

Finally, if the measurement is unread, we find $\hat{\rho}\,' = \sum_m P_m \times \hat{\rho}'_{|m}$. Combining Eqs. (57) and (59), it yields

$$\boxed{\hat{\rho}\,' = \sum_m \hat{M}_m \, \hat{\rho} \, \hat{M}_m^\dagger} \ . \tag{60}$$

Of course, the conditional and unconditional operators $\hat{\rho}'_{|m}$ [Eq. (57)] and $\hat{\rho}'$ [Eq. (60)] fulfill all the necessary conditions for being legitimate density matrices, see exercise 1c, page 21.

Equation (60) is of utmost importance. While it has been found here as a result of an unread measurement, we shall see later that it is, more generally, the evolution equation of an open system coupled to an environment.

3.4 Use cases of generalized measurements

We finally briefly discuss some use cases of POVMs. As discussed above, many measurements on individual quantum systems cannot be described by PVMs. Instead, POVMs provide us with a suitable framework for describing real measurement processes. For instance, consider again the measurement of an atomic state by position-dependent ionization, as discussed in the introduction of Sec. 3. It can be checked that the operators $\hat{M}_{\mathrm{g}} = |\mathrm{ion}\rangle\langle \mathrm{g}|$ and $\hat{M}_{\mathrm{e}} = |\mathrm{ion}\rangle\langle \mathrm{e}|$ introduced in Eq. (39) are legitimate Kraus operators. Assume that the atom is in any superposition state $|\psi\rangle = \alpha\,|\mathrm{g}\rangle + \beta\,|\mathrm{e}\rangle$. The atom is supposedly measured in the state $|\mathrm{g}\rangle$ with probability $|\alpha|^2$ and in the state $|\mathrm{e}\rangle$ with probability $|\beta|^2$. This is indeed what is found using the POVM formula of Eq. (48): $P_{\mathrm{g}} = |\hat{M}_{\mathrm{g}}\,|\psi\rangle\,|^2 = |\alpha|^2$ and $P_{\mathrm{e}} = |\hat{M}_{\mathrm{e}}\,|\psi\rangle\,|^2 = |\beta|^2$. Moreover, whatever the measurement result, the atom is ionized after the measurement, which is well described by the formula (46):

$$|\psi'\rangle = \frac{\hat{M}_{\mathrm{g/e}}|\psi\rangle}{\left|\hat{M}_{\mathrm{g/e}}|\psi\rangle\right|} = |\mathrm{ion}\rangle \ . \tag{61}$$

Finally, it is straightforward to check that the operators $\hat{M}_{\mathrm{g}}$ and $\hat{M}_{\mathrm{e}}$ satisfy the completeness relation (44).

This kind of applications is, however, limited in that, if we are not interested in the system state after the measurement, the measurement results would be just as those given by standard PVMs. There are in fact many other applications of POVMs. In general, POVMs only provide partial information about the state of the measured system. This is due to the fact that the after-measurement states are generally non-orthogonal. Hence, being in one does not exclude being in the other. An example is provided by the QND measurement in CQED as discussed in Sec. 2.4. Assume that we do not apply the cavity R_2. In this case, the final atom-cavity state reads as Eq. (36). If the atom is measured in $|\mathrm{g}\rangle$, the cavity field is projected onto $-i\alpha\,|0\rangle + \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle$; If it is measured in $|\mathrm{e}\rangle$, the cavity field is projected onto $-i\alpha\,|0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle$. In general, these cavity states are not orthogonal (In fact, they are orthogonal if and only if $|\alpha|^2 = |\beta|^2 = 1/2$). This case is analyzed in detail in problem B.1 on page 21.

Nevertheless, while POVMs only provide partial information on the state of the system, one can build many more Kraus operators – as relevant to POVMs – than projectors – as relevant to PVMs. Indeed, a PVM can only return a number of different results at most equal to the dimension of the Hilbert space of the measured system. On the contrary, a POVM can return a number of different results up to the dimension of the Hilbert space of the ancilla (or the meter). This allows to accumulate much more information using an ancilla larger than the system of interest. An application is the discrimation of quantum states as treated in the problem B.2 on page 23.

A Exercise: After-measurement density matrix

1. (a) Under what conditions is an operator $\hat{\rho}$ a density matrix?
   (b) Show that the operator $\hat{\rho}'$ in Eq. (6) fulfills all the necessary conditions for being a density matrix.
   (c) Show that the conditional and unconditional operators $\hat{\rho}'_{|m}$ and $\hat{\rho}'$ associated to a POVM, Eqs. (57) and (60), fulfill all the necessary conditions for being density matrices.
2. Derive Eq. (8), and show that one recovers the usual Born rule, Eq. (1), in the case of a pure state.
3. Prove Eq. (18) i.e., for two systems A and B, $P_j = \left|\big(\hat{\mathcal{P}}_{j,A} \otimes \hat{\mathbb{1}}_B\big)|\Psi\rangle_{A\otimes B}\right|^2 = \mathrm{Tr}\big(\hat{\mathcal{P}}_j \hat{\rho}_A\big)$ for the bipartite state $|\Psi\rangle_{A\otimes B} = \sum_n c_n\, |\psi_n\rangle_A \otimes |\chi_n\rangle_B$ where the $|\chi_n\rangle$'s form an orthonormal basis of B.

B Problems

B.1 POVMs in cavity quantum electrodynamics

Consider the cavity quantum electrodynamics experimental device discussed in Sec. 2.4, see also Fig. 5. The purpose of this apparatus is to realize a QND measurement of the radiation field state in cavity C using a two-level atom A with ground and excited states $|\mathrm{g}\rangle$ and $|\mathrm{e}\rangle$, respectively. Here we aim at finding the Kraus operators $\hat{M}_\mathrm{g}$ and $\hat{M}_\mathrm{e}$ associated to the measurement of the atom in either state $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$. With respect to the vocabulary of Sec. 3, the cavity mode plays the role of the system of interest S and the atom that of the meter M. The cavity mode is initially in the arbitrary state $|\psi\rangle_C = \alpha\,|0\rangle + \beta\,|1\rangle$ and the reference state of the atom (meter) is $|\mathrm{e}\rangle$.

B.1.1 Two-cavity apparatus

Consider first the apparatus restricted to only two cavities, i.e. ignore cavity R_2. We have shown in Sec. 2.4 that the atom-cavity state at the output of cavity C reads as

$$|\Psi''\rangle_{A\otimes C} = -i\alpha\frac{|\mathrm{g}\rangle + |\mathrm{e}\rangle}{\sqrt{2}} \otimes |0\rangle + \beta \mathrm{e}^{-i\omega T_C}\frac{|\mathrm{g}\rangle - |\mathrm{e}\rangle}{\sqrt{2}} \otimes |1\rangle \ , \tag{62}$$

see Eq. (36).

1. Find the expressions of $\hat{M}_{\mathrm{g}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)$ and $\hat{M}_{\mathrm{e}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)$.

   Hint: Write the atom-cavity state (62) in the form $|\Psi\rangle = |\psi_{\mathrm{g}}\rangle_C \otimes |\mathrm{g}\rangle_A + |\psi_{\mathrm{e}}\rangle_C \otimes |\mathrm{e}\rangle_A$ and identify $\hat{M}_{\mathrm{g/e}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)$ as in Eq. (42).

2. Deduce the expressions of the Kraus operators $\hat{M}_{\mathrm{g}}$ and $\hat{M}_{\mathrm{e}}$.

   Hint: Write the action of these operators onto the basis states $|0\rangle$ and $|1\rangle$.

3. Check the completeness relation, $\hat{M}_{\mathrm{g}}^\dagger\hat{M}_{\mathrm{g}} + \hat{M}_{\mathrm{e}}^\dagger\hat{M}_{\mathrm{e}} = \hat{\mathbb{1}}$. Are these Kraus operators Hermitian?

4. Compute the probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the measured atomic state, $P_{0|\mathrm{g}}$, $P_{0|\mathrm{e}}$, $P_{1|\mathrm{g}}$, and $P_{1|\mathrm{e}}$.

5. Compute the probabilities P_{g} and P_{e} of measuring the atom in $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$. Does the measurement of the atomic state yield information about the photon number?

B.1.2 Three-cavity apparatus

Consider now the complete three-cavity problem, including cavity R_2. We recall that we found in Sec. 2.4 that the atom-cavity state at the output of cavity R_2 reads as

$$|\Psi'''\rangle_{A\otimes C} = -i\alpha\,|\mathrm{g}\rangle \otimes |0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|\mathrm{e}\rangle \otimes |1\rangle \ , \tag{63}$$

see Eq. (38).

6. Find the expressions of the operators $\hat{M}_{\mathrm{g}}$ and $\hat{M}_{\mathrm{e}}$, and check the completeness relation. Are these operators Hermitian?

7. Compute the probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the measured atomic state, $P_{0|\mathrm{g}}$, $P_{0|\mathrm{e}}$, $P_{1|\mathrm{g}}$, and $P_{1|\mathrm{e}}$.

8. Compute the probabilities P_{g} and P_{e} of measuring the atom in $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$. Does the measurement of the atomic state yield information about the photon number?

B.1.3 Incomplete measurement

We now assume that the third cavity realizes and incomplete $\pi/2$ rotation, corresponding to the transformation

$$\frac{|\mathrm{g}\rangle + |\mathrm{e}\rangle}{\sqrt{2}} \longrightarrow u\,|\mathrm{g}\rangle + v\,|\mathrm{e}\rangle \qquad \text{and} \qquad \frac{|\mathrm{g}\rangle - |\mathrm{e}\rangle}{\sqrt{2}} \longrightarrow v^*\,|\mathrm{g}\rangle - u^*\,|\mathrm{e}\rangle\,, \tag{64}$$

where $u, v \in \mathbb{C}$ and $|u|^2 + |v|^2 = 1$.

9. Justify that it corresponds to a rotation on the Bloch sphere. It is not required to determine which.
10. Write the atom-cavity state $|\Psi'''\rangle_{A\otimes C}$ after R_2 in the form of Eq. (42). Deduce the expressions of the Kraus operators $\hat{M}_\mathrm{g}$ and $\hat{M}_\mathrm{e}$, and check the completeness relation.
11. Compute the probabilities of measuring the atom in $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$ as a function of $P_0 = |\alpha|^2$ and $\epsilon = |u|^2$. Plot P_g as a function of P_0 for various values of $\epsilon \in [1/2, 1]$.
12. Justify that $\mathcal{E} = |2\epsilon - 1|$ may be called the *efficiency* of the QND measurement. Why do we restrict ourselves to $\epsilon =\in [1/2, 1]$? What would be smart to do for $\epsilon \in [0, 1/2]$?

B.2 Discriminating quantum states: PVM versus POVM

Alice prepares a qubit in either states $|\psi\rangle$ or $|\phi\rangle$, each with probability 1/2. These two states are represented by the Bloch vectors $\boldsymbol{\psi}$ or $\boldsymbol{\phi}$, which make an angle θ with each other. Bob aims at determining *with absolute certainty* the state prepared by Alice.

1. Assume Bob makes a PVM on the qubit along a vector $\boldsymbol{u}$.
   (a) With what probability can he determine the state of the qubit with certainty if he takes $\boldsymbol{u}$ different from $\pm\boldsymbol{\psi}$ and $\pm\boldsymbol{\phi}$?

   Hint : We recall that the only information we can get about the initial state of a quantum system is that it cannot be orthogonal to the measurement result.

   (b) Same question if Bob chooses $\boldsymbol{u} = -\boldsymbol{\psi}$ or $-\boldsymbol{\phi}$. How much is it for $\theta = 120°$?
2. Assume now that the system is a 1/2 spin and that Bob makes a POVM determined by the Kraus operators

$$\hat{M}_m = \sqrt{\frac{\hat{\mathbb{1}} + \boldsymbol{u}_m \cdot \hat{\boldsymbol{\sigma}}}{3}}\,, \tag{65}$$

with $\hat{\boldsymbol{\sigma}}$ the spin vector operator, $\boldsymbol{u}_1 = -\boldsymbol{\psi}$, $\boldsymbol{u}_2 = -\boldsymbol{\phi}$, and $\boldsymbol{u}_3 = \boldsymbol{\psi} + \boldsymbol{\phi}$. We assume here $\theta = 120°$, so that the $\boldsymbol{u}_m$'s point towards the vertices of an equilateral triangle on the Bloch sphere.

(a) Check the completeness relation.

(b) Show that if Bob finds $m = 1$ or 2, one of the states $|\psi\rangle$ or $|\phi\rangle$ can be excluded.

(c) Deduce that Bob can determine the qubit state with probability $P = 1/2$. Comment.

References

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