A Exercise: After-measurement density matrix

1. (a) Under what conditions is an operator $\hat{\rho}$ a density matrix?

   Solution: An operator $\hat{\rho}$ can be interpreted as a density matrix iff and only if it is Hermitian, semi-definite positive, and of unit trace.

   (b) Show that the operator $\hat{\rho}\,'$ in Eq. (6) fulfills all the necessary conditions for being a density matrix.

   Solution: The operator $\hat{\rho}\,'$ in Eq. (6),

$$\hat{\rho}\,' = \sum_j \hat{\mathcal{P}}_j |\psi\rangle\langle\psi|\hat{\mathcal{P}}_j \ ,$$

   is clearly Hermitian since projectors are Hermitian operators. It is semi-definite positive since, for any $|\phi\rangle$,

$$\langle\phi|\,\hat{\rho}\,|\phi\rangle = \sum_j \langle\phi|\,\hat{\mathcal{P}}_j|\psi\rangle\langle\psi|\hat{\mathcal{P}}_j\,|\phi\rangle = \sum_j \left|\langle\psi|\hat{\mathcal{P}}_j\,|\phi\rangle\right|^2 \geq 0 \ .$$

   Finally, it is of unit trace: Using an orthonormal basis $\{|\alpha\rangle\}$, we find

$$\mathrm{Tr}(\hat{\rho}\,') = \sum_\alpha \sum_j \langle\alpha|\hat{\mathcal{P}}_j\,|\psi\rangle\langle\psi|\,\hat{\mathcal{P}}_j|\alpha\rangle = \sum_j \sum_\alpha \langle\psi|\hat{\mathcal{P}}_j\,|\alpha\rangle\langle\alpha|\,\hat{\mathcal{P}}_j|\psi\rangle \ ,$$

   where we use the completeness relation $\sum_\alpha |\alpha\rangle\langle\alpha| = \hat{\mathbb{1}}$. Then, using the idempotent property and the completeness relation of projectors, $\hat{\mathcal{P}}_j^2 = \hat{\mathcal{P}}_j$ and $\sum_j \hat{\mathcal{P}}_j = \hat{\mathbb{1}}$, we find

$$\mathrm{Tr}(\hat{\rho}\,') = \langle\psi|\psi\rangle = 1 \ .$$

   (c) Show that the conditional and unconditional operators $\hat{\rho}'_{|m}$ and $\hat{\rho}\,'$ associated to a POVM, Eqs. (57) and (60), fulfill all the necessary conditions for being density matrices.

   Solution: Let us start with $\hat{\rho}'_{|m}$,

$$\hat{\rho}'_{|m} = \frac{\hat{M}_m\,\hat{\rho}\,\hat{M}_m^\dagger}{\mathrm{Tr}\big(\hat{M}_m\,\hat{\rho}\,\hat{M}_m^\dagger\big)} \ .$$

   The Hermitian and unit trace characters are obvious. Moreover, for any state $|\phi\rangle$, we have $\langle\phi|\,\hat{M}_m\hat{\rho}\hat{M}_m^\dagger\,|\phi\rangle \geq 0$ since $\hat{M}_m^\dagger\,|\phi\rangle$ is a ket and $\hat{\rho}$ is semi-definite positive. For the same reason $\mathrm{Tr}\big(\hat{M}_m\,\hat{\rho}\,\hat{M}_m^\dagger\big) \geq 0$, and we find that $\hat{\rho}'_{|m}$ is semi-definite positive.

   Now, we have $\hat{\rho}\,' = \sum_m P_m \times \hat{\rho}'_{|m}$, where P_m is the probability of measuring m. Since probabilities are real, nonnegative numbers and normalized by $\sum_m P_m = 1$, we immediately find that $\hat{\rho}\,'$ is a density matrix.

2. Derive Eq. (8), and show that one recovers the usual Born rule, Eq. (1), in the case of a pure state.

   Solution: We have

$$\begin{aligned} P_j &= \sum_n \Pi_n P_{j|n} = \sum_n \Pi_n |\langle j|\psi_n\rangle|^2 = \sum_n \Pi_n \langle j\,|\psi_n\rangle\langle\psi_n|\,j\rangle = \langle j|\hat{\rho}|j\rangle \\ &= \mathrm{Tr}\big(\hat{\rho}\,|j\rangle\langle j|\,\big) = \mathrm{Tr}\big(\hat{\rho}\,\hat{\mathcal{P}}_j\big) \ . \end{aligned}$$

3. Prove Eq. (18) i.e., for two systems A and B, $P_j = \big|\big(\hat{\mathcal{P}}_{j,A}\otimes\hat{\mathbb{1}}_B\big)|\Psi\rangle_{A\otimes B}\big|^2 = \mathrm{Tr}\big(\hat{\mathcal{P}}_j\hat{\rho}_A\big)$ for the bipartite state $|\Psi\rangle_{A\otimes B} = \sum_n c_n\,|\psi_n\rangle_A\otimes|\chi_n\rangle_B$ where the $|\chi_n\rangle$'s form an orthonormal basis of B.

   Solution: Using the general result for a PVM on $A\otimes B$, we find

$$\begin{aligned} P_j &= \big|\big(\hat{\mathcal{P}}_{j,A}\otimes\hat{\mathbb{1}}_B\big)|\Psi\rangle\big|^2 = \Big|\sum_n c_n\left(\hat{\mathcal{P}}_j|\psi_n\rangle\right)_A\otimes|\chi_n\rangle_B\Big|^2 \\ &= \sum_n \langle\psi_n|\,\hat{\mathcal{P}}_j\times|c_n|^2\times\hat{\mathcal{P}}_j\,|\psi_n\rangle = \mathrm{Tr}\Big(\hat{\mathcal{P}}_j\sum_n |c_n|^2\,|\psi_n\rangle\langle\psi_n|\Big) = \mathrm{Tr}\big(\hat{\mathcal{P}}_j\hat{\rho}_A\big) \ . \end{aligned}$$

B Problems

B.1 POVMs in cavity quantum electrodynamics

Consider the cavity quantum electrodynamics experimental device discussed in Sec. 2.4, see also Fig. 5. The purpose of this apparatus is to realize a QND measurement of the radiation field state in cavity C using a two-level atom A with ground and excited states $|\mathrm{g}\rangle$ and $|\mathrm{e}\rangle$, respectively. Here we aim at finding the Kraus operators $\hat{M}_\mathrm{g}$ and $\hat{M}_\mathrm{e}$ associated to the measurement of the atom in either state $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$. With respect to the vocabulary of Sec. 3, the cavity mode plays the role of the system of interest S and the atom that of the meter M. The cavity mode is initially in the arbitrary state $|\psi\rangle_C = \alpha\,|0\rangle + \beta\,|1\rangle$ and the reference state of the atom (meter) is $|\mathrm{e}\rangle$.

B.2 Two-cavity apparatus

Consider first the apparatus restricted to only two cavities, i.e. ignore cavity R_2. We have shown in Sec. 2.4 that the atom-cavity state at the output of cavity C reads as

$$|\Psi''\rangle_{A\otimes C} = -i\alpha\frac{|\mathrm{g}\rangle+|\mathrm{e}\rangle}{\sqrt{2}}\otimes|0\rangle + \beta\mathrm{e}^{-i\omega T_C}\frac{|\mathrm{g}\rangle-|\mathrm{e}\rangle}{\sqrt{2}}\otimes|1\rangle \ , \quad (62)$$

see Eq. (36).

1. Find the expressions of $\hat{M}_{\mathrm{g}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)$ and $\hat{M}_{\mathrm{e}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)$.

   Hint: Write the atom-cavity state (62) in the form $|\Psi\rangle = |\psi_{\mathrm{g}}\rangle_C \otimes |\mathrm{g}\rangle_A + |\psi_{\mathrm{e}}\rangle_C \otimes |\mathrm{e}\rangle_A$ and identify $\hat{M}_{\mathrm{g/e}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)$ as in Eq. (42).

   Solution: Starting from Eq. (62), we find

   $$|\Psi''\rangle_{A\otimes C} = \frac{1}{\sqrt{2}}\Big(-i\alpha\,|0\rangle + \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle\,\Big) \otimes |\mathrm{g}\rangle + \frac{1}{\sqrt{2}}\Big(-i\alpha\,|0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle\,\Big) \otimes |\mathrm{e}\rangle \;.$$

   According to Eq. (42), we then find

   $$\hat{M}_{\mathrm{g}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big) = \frac{1}{\sqrt{2}}\Big(-i\alpha\,|0\rangle + \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle\,\Big)$$

   and

   $$\hat{M}_{\mathrm{e}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big) = \frac{1}{\sqrt{2}}\Big(-i\alpha\,|0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle\,\Big) \;.$$

2. Deduce the expressions of the Kraus operators $\hat{M}_{\mathrm{g}}$ and $\hat{M}_{\mathrm{e}}$.

   Hint: Write the action of these operators onto the basis states $|0\rangle$ and $|1\rangle$.

   Solution: The action of $\hat{M}_{\mathrm{g/e}}$ on the basis states are found by setting, on the one hand, $\alpha = 1$ and $\beta = 0$ and, on the other hand, $\alpha = 0$ and $\beta = 1$. It yields

   $$\hat{M}_{\mathrm{g/e}}\,|0\rangle = \frac{-i}{\sqrt{2}}\,|0\rangle \qquad \text{and} \qquad \hat{M}_{\mathrm{g/e}}\,|1\rangle = \pm\frac{\mathrm{e}^{-i\omega T_C}}{\sqrt{2}}\,|1\rangle$$

   The Kraus operators are then diagonal and read as

   $$\hat{M}_{\mathrm{g/e}} = \frac{-i}{\sqrt{2}}\,|0\rangle\,\langle 0| \pm \frac{\mathrm{e}^{-i\omega T_C}}{\sqrt{2}}\,|1\rangle\,\langle 1| \;.$$

3. Check the completeness relation, $\hat{M}_{\mathrm{g}}^\dagger \hat{M}_{\mathrm{g}} + \hat{M}_{\mathrm{e}}^\dagger \hat{M}_{\mathrm{e}} = \hat{\mathbb{1}}$. Are these Kraus operators Hermitian?

   Solution: We have

   $$\hat{M}_{\mathrm{g/e}}^\dagger = \frac{+i}{\sqrt{2}}\,|0\rangle\,\langle 0| \pm \frac{\mathrm{e}^{+i\omega T_C}}{\sqrt{2}}\,|1\rangle\,\langle 1| \;.$$

   These Kraus operators are thus not Hermitian. Moreover, we find $\hat{M}_{\mathrm{g/e}}^\dagger \hat{M}_{\mathrm{g/e}} = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1| = \frac{1}{2}\hat{\mathbb{1}}$, and $\hat{M}_{\mathrm{g}}^\dagger \hat{M}_{\mathrm{g}} + \hat{M}_{\mathrm{e}}^\dagger \hat{M}_{\mathrm{e}} = \hat{\mathbb{1}}$.

4. Compute the probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the measured atomic state, $P_{0|\mathrm{g}}$, $P_{0|\mathrm{e}}$, $P_{1|\mathrm{g}}$, and $P_{1|\mathrm{e}}$.

   Solution: If the atom is measured in g, the after-measurement state of the cavity mode is the normalized state $|\psi'_{|\mathrm{g}}\rangle \propto \hat{M}_{\mathrm{g}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)$, i.e., according to the result of question 1,
   $$|\psi'_{|\mathrm{g}}\rangle = -i\alpha\,|0\rangle + \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle \ .$$
   The probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the atom be measured in g are thus
   $$P_{0|\mathrm{g}} = |\alpha|^2 \qquad \text{and} \qquad P_{1|\mathrm{g}} = |\beta|^2 \ .$$
   Similarly, if the atom is measured in e, the after-measurement state of the cavity mode is the normalized state $|\psi'_{|\mathrm{e}}\rangle \propto \hat{M}_{\mathrm{e}}\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)$, i.e.
   $$|\psi'_{|\mathrm{e}}\rangle = -i\alpha\,|0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|1\rangle \,,$$
   and the conditional proabilities are again
   $$P_{0|\mathrm{e}} = |\alpha|^2 \qquad \text{and} \qquad P_{1|\mathrm{e}} = |\beta|^2 \ .$$
   These probabilites are thus independent of the measurement result on the atom.

5. Compute the probabilities P_{g} and P_{e} of measuring the atom in $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$. Does the measurement of the atomic state yield information about the photon number?

   Solution: The probabilities of measuring the atom either in $|\mathrm{g}\rangle$ or in $|\mathrm{e}\rangle$ are given by $P_{\mathrm{g/e}} = |\hat{M}_{\mathrm{g/e}}\,|\psi\rangle_C\,|^2$, and, using the result of question 1, we find i.e.
   $$P_{\mathrm{g}} = P_{\mathrm{e}} = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2} \ .$$
   These probabilities are thus completely independent of the photon state in the cavity. We thus don't get any information about the cavity state by measuring the atomic state in this configuration.

B.3 Three-cavity apparatus

Consider now the complete three-cavity problem, including cavity R_2. We recall that we found in Sec. 2.4 that the atom-cavity state at the output of cavity R_2 reads as

$$|\Psi'''\rangle_{A\otimes C} = -i\alpha\,|\mathrm{g}\rangle \otimes |0\rangle - \beta \mathrm{e}^{-i\omega T_C}\,|\mathrm{e}\rangle \otimes |1\rangle \ \ , \tag{63}$$

see Eq. (38).

6. Find the expressions of the operators $\hat{M}_\mathrm{g}$ and $\hat{M}_\mathrm{e}$, and check the completeness relation. Are these operators Hermitian?

   Solution: We proceed similarly as above. The atom-cavity state is now directly written as in Eq. (42), and we directly read

   $$\hat{M}_\mathrm{g}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big) = -i\alpha\,|0\rangle \qquad \text{and} \qquad \hat{M}_\mathrm{e}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big) = -\beta \mathrm{e}^{-i\omega T_C}\,|1\rangle \ .$$

   We thus find $\hat{M}_\mathrm{g}\,|0\rangle = -i\,|0\rangle$, $\hat{M}_\mathrm{g}\,|1\rangle = 0$, $\hat{M}_\mathrm{e}\,|0\rangle = 0$, and $\hat{M}_\mathrm{e}\,|1\rangle = -\mathrm{e}^{-i\omega T_C}\,|1\rangle$, i.e.

   $$\hat{M}_\mathrm{g} = -i\,|0\rangle\langle 0| \qquad \text{and} \qquad \hat{M}_\mathrm{e} = -\mathrm{e}^{-i\omega T_C}\,|1\rangle\langle 1| \ .$$

   Clearly, these operators are not Hermitian (except $\hat{M}_\mathrm{e}$ for $\omega T_C \equiv 0\ [\pi]$). Moreover, we find $\hat{M}_\mathrm{g}^\dagger \hat{M}_\mathrm{g} + \hat{M}_\mathrm{e}^\dagger \hat{M}_\mathrm{e} = |0\rangle\langle 0| + |1\rangle\langle 1| = \hat{\mathbb{1}}$. In fact, we find here that the Kraus operators are, up to a phase factor, the projectors onto $|0\rangle$ and $|1\rangle$, respectively.

7. Compute the probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the measured atomic state, $P_{0|\mathrm{g}}$, $P_{0|\mathrm{e}}$, $P_{1|\mathrm{g}}$, and $P_{1|\mathrm{e}}$.

   Solution: If the atom is measured in g, the after-measurement state of the cavity mode is

   $$|\psi'_{|\mathrm{g}}\rangle = \frac{\hat{M}_\mathrm{g}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)}{\Big|\hat{M}_\mathrm{g}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)\Big|} = \mathrm{e}^{i\theta}\,|0\rangle\,,$$

   where $\mathrm{e}^{i\theta} = -i\alpha/|\alpha|$. The probabilities that the cavity has either $n = 0$ or $n = 1$ photon, conditional to the atom be measured in g are thus

   $$P_{0|\mathrm{g}} = 1 \qquad \text{and} \qquad P_{1|\mathrm{g}} = 0 \ .$$

   Similarly, if the atom is measured in e, the after-measurement state of the cavity mode is

   $$|\psi'_{|\mathrm{e}}\rangle = \frac{\hat{M}_\mathrm{e}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)}{\Big|\hat{M}_\mathrm{e}\big(\alpha\,|0\rangle + \beta\,|1\rangle\big)\Big|} = \mathrm{e}^{i\theta'}\,|1\rangle \ ,$$

   where $\mathrm{e}^{i\theta'} = -\mathrm{e}^{-i\omega T_C}\beta/|\beta|$, and the conditional probabilities are

   $$P_{0|\mathrm{e}} = 0 \qquad \text{and} \qquad P_{1|\mathrm{e}} = 1 \ .$$

   Hence, in this configuration with three cavities, the photon and atom states are perfectly correlated.

8. Compute the probabilities P_g and P_e of measuring the atom in $|\mathrm{g}\rangle$ or $|\mathrm{e}\rangle$. Does the measurement of the atomic state yield information about the photon number?

   Solution: The probabilities of measuring the atom either in $|\mathrm{g}\rangle$ or in $|\mathrm{e}\rangle$ are given by $P_{\mathrm{g/e}} = |\hat{M}_{\mathrm{g/e}}\,|\psi\rangle_C\,|^2$, and we find

   $$P_\mathrm{g} = |\alpha|^2 \qquad \text{and} \qquad P_\mathrm{e} = |\beta|^2 \ .$$

Therefore the probabilities of measuring the atom in either g or e are now equal to the probabilities of having 0 or 1 photon in the initial state of the cavity. Hence the three-cavity apparatus realizes a perfect QND measurement of the photon number in the cavity.

B.4 Incomplete measurement

We now assume that the third cavity realizes and incomplete $\pi/2$ rotation, corresponding to the transformation

$$\frac{|\mathrm{g}\rangle + |\mathrm{e}\rangle}{\sqrt{2}} \longrightarrow u\,|\mathrm{g}\rangle + v\,|\mathrm{e}\rangle \qquad \text{and} \qquad \frac{|\mathrm{g}\rangle - |\mathrm{e}\rangle}{\sqrt{2}} \longrightarrow v^*\,|\mathrm{g}\rangle - u^*\,|\mathrm{e}\rangle\,, \tag{64}$$

where $u, v \in \mathbb{C}$ and $|u|^2 + |v|^2 = 1$.

9. Justify that it corresponds to a rotation on the Bloch sphere. It is not required to determine which.

   Solution: Here we are dealing with a two-level system. Its state may be represented by a unit vector on the Bloch sphere and any unitary operation corresponds to a rotation. The proposed transformation is clearly unitary since it transforms an orthonormal basis into another orthonormal basis.

   The ideal case as discussed in Sec. B.3 is recovered for $u = 1$ and $v = 0$.

10. Write the atom-cavity state $|\Psi'''\rangle_{A\otimes C}$ after R_2 in the form of Eq. (42). Deduce the expressions of the Kraus operators $\hat{M}_\mathrm{g}$ and $\hat{M}_\mathrm{e}$, and check the completeness relation.

    Solution: The atom-cavity state after cavity C is given by Eq. (62). Then, the cavity R_2 only acts on the atomic state, yet we have to keep in mind that the cavity state evolves freely. Applying the transformation (64) to the atomic state and the transformation $|n\rangle \longrightarrow \mathrm{e}^{-i\omega T_R}\,|n\rangle$ to the cavity state, we find

    $$|\Psi''\rangle_{A\otimes C} = -i\alpha\big(u\,|\mathrm{g}\rangle + v\,|\mathrm{e}\rangle\big) \otimes |0\rangle + \beta'\big(v^*\,|\mathrm{g}\rangle - u^*\,|\mathrm{e}\rangle\big) \otimes |1\rangle \ \ ,$$

    where $\beta' = \beta\mathrm{e}^{-i\omega(T_C+T_R)}$. It yields

    $$|\Psi''\rangle_{A\otimes C} = \big(-i\alpha u\,|0\rangle + \beta' v^*\,|1\rangle\big) \otimes |\mathrm{g}\rangle + \big(-i\alpha v\,|0\rangle - \beta' u^*\,|1\rangle\big) \otimes |\mathrm{e}\rangle \ \ .$$

    The corresponding Kraus operators are thus

    $$\hat{M}_\mathrm{g} = -iu\,|0\rangle\langle 0| + v^*\mathrm{e}^{i\theta'}\,|1\rangle\langle 1| \qquad \text{and} \qquad \hat{M}_\mathrm{e} = -iv\,|0\rangle\langle 0| - u^*\mathrm{e}^{i\theta'}\,|1\rangle\langle 1| \ \ ,$$

    where $\theta' = -\omega(T_C + T_R)$. Moreover, we find

    $$\hat{M}_\mathrm{g}^\dagger\hat{M}_\mathrm{g} + \hat{M}_\mathrm{e}^\dagger\hat{M}_\mathrm{e} = |u|^2\,|0\rangle\langle 0| + |v|^2\,|1\rangle\langle 1| + |v|^2\,|0\rangle\langle 0| + |u|^2\,|1\rangle\langle 1| = \hat{\mathbb{1}}\ ,$$

    and the completeness relation is fulfilled.

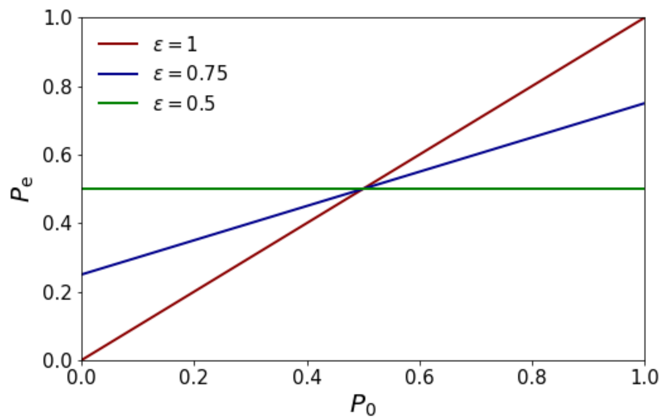
11. Compute the probabilities of measuring the atom in $|g\rangle$ or $|e\rangle$ as a function of $P_0 = |\alpha|^2$ and $\epsilon = |u|^2$. Plot P_g as a function of P_0 for various values of $\epsilon \in [1/2, 1]$.

Solution: The probability of measuring the atom in the ground state is $P_g = |\hat{M}_g |\psi\rangle_C|^2$, where $|\psi\rangle_C = \alpha |0\rangle + \beta |1\rangle$ is the initial cavity state. It yields

$$P_g = |-i\alpha u |0\rangle + \beta' v^* |1\rangle|^2 = |\alpha|^2 \cdot |u|^2 + |\beta|^2 \cdot |v^*|^2 = \epsilon P_0 + (1-\epsilon)(1-P_0) ,$$

that is

$$P_g = P_0 + (1-\epsilon)(1-2P_0) = (1-\epsilon) + (2\epsilon - 1)P_0 .$$



12. Justify that $\mathcal{E} = |2\epsilon - 1|$ may be called the *efficiency* of the QND measurement. Why do we restrict ourselves to $\epsilon =\in [1/2, 1]$? What would be smart to do for $\epsilon \in [0, 1/2]$?

Solution: The QND measurement aims at measuring the photon number via the atomic state. In the ideal case, the correspondence woud be

$$|g\rangle \longleftrightarrow |0\rangle \qquad \text{and} \qquad |e\rangle \longleftrightarrow |1\rangle .$$

The probablity of measuring P_g yields a fair estimate of P_0 with a relative error on the slope of $|2\epsilon - 1| = \mathcal{E}$. It can thus be seen as the efficiency of the QND measurement.

For $\epsilon =\in [0, 1/2]$, it is better to invert the state correspondence and use

$$|e\rangle \longleftrightarrow |0\rangle \qquad \text{and} \qquad |g\rangle \longleftrightarrow |1\rangle .$$

We then use $\epsilon' = 1 - \epsilon$ and the probability of finding the atom in e reads as

$$P_e = 1 - (1-\epsilon) - (2\epsilon - 1)P_0 = (1-\epsilon') + (2\epsilon' - 1)P_0 .$$

We then recover the same efficiency as above, $\mathcal{E} = |2\epsilon' - 1| = |2\epsilon - 1|$.

B.5 Discriminating quantum states: PVM versus POVM

Alice prepares a qubit in either states $|\psi\rangle$ or $|\phi\rangle$, each with probability 1/2. These two states are represented by the Bloch vectors $\boldsymbol{\psi}$ or $\boldsymbol{\phi}$, which make an angle θ with each other. Bob aims at determining *with absolute certainty* the state prepared by Alice.

1. Assume Bob makes a PVM on the qubit along a vector $\boldsymbol{u}$ different from $\pm\boldsymbol{\psi}$ and $\pm\boldsymbol{\phi}$.

   (a) With what probability can he determine the state of the qubit with certainty?

   Hint : We recall that the only information we can get about the initial state of a quantum system is that it cannot be orthogonal to the measurement result.
   Solution: If Bob measures along $\boldsymbol{u}$ different from $\pm\boldsymbol{\psi}$ and $\pm\boldsymbol{\phi}$, he can get ± 1 with nonzero probabilities and he can never conclude with absolute certainty.

   (b) Same question if Bob chooses $\boldsymbol{u} = -\boldsymbol{\psi}$ or $-\boldsymbol{\phi}$. How much is it for $\theta = 120°$?

   Solution: Assume Bob measures along $-\boldsymbol{\psi}$ and finds $+1$, he can be sure that the state was not $\boldsymbol{\psi}$ (which is associated to a ket orthogonal to that associated to $-\boldsymbol{\psi}$) and he can safely conclude that the state prepared by Alice was $\boldsymbol{\phi}$. It is the only case in which he can conclude and it occurs with probability

$$P = \underbrace{\frac{1}{2}}_{\text{prob. Alice prepares } \boldsymbol{\phi}} \times \underbrace{\frac{1 - \boldsymbol{\psi}\cdot\boldsymbol{\phi}}{2}}_{\text{prob. of then finding } +1} ,$$

   that is

$$P(\theta) = \frac{1-\cos\theta}{4} .$$

   For $\theta = 2\pi/3$ (120°), we find $P(2\pi/3) = 3/8 = 0.375$.
   By symmetry, we obtain a similar result if Bob decides to measure along $-\boldsymbol{\phi}$.

2. Assume now that the system is a 1/2 spin and that Bob makes a POVM determined by the Kraus operators

$$\hat{M}_m = \sqrt{\frac{\hat{\mathbb{1}} + \boldsymbol{u}_m\cdot\hat{\boldsymbol{\sigma}}}{3}} , \qquad (65)$$

   with $\hat{\boldsymbol{\sigma}}$ the spin vector operator, $\boldsymbol{u}_1 = -\boldsymbol{\psi}$, $\boldsymbol{u}_2 = -\boldsymbol{\phi}$, and $\boldsymbol{u}_3 = \boldsymbol{\psi} + \boldsymbol{\phi}$. We assume here $\theta = 120°$, so that the $\boldsymbol{u}_m$'s point towards the vertices of an equilateral triangle on the Bloch sphere.

   (a) Check the completeness relation.

   Solution: We have

$$\sum_{m=1}^{3} \hat{M}_m^\dagger \hat{M}_m = \sum_{m=1}^{3} \frac{\hat{\mathbb{1}} + \boldsymbol{u}_m\cdot\hat{\boldsymbol{\sigma}}}{3} = \hat{\mathbb{1}}$$

since $\sum_{m=1}^{3} \boldsymbol{u}_m = 0$.

(b) Show that if Bob finds $m = 1$ or 2, one of the states $|\psi\rangle$ or $|\phi\rangle$ can be excluded.
Solution: If Bob finds $m = 1$, then $|\psi\rangle$ is excuded. If he finds $m = 2$, then $|\phi\rangle$ is excluded. If he finds $m = 3$, he cannot conclude.

(c) Deduce that Bob can determine the qubit state with probability $P = 1/2$. Comment.
Solution: The probability to finding a conclusive result is

$$P = P_{A:\psi} \times P_{B:2|\psi} + P_{A:\phi} \times P_{B:1|\phi} \ .$$

Here we have $P_{A:\psi} = P_{A:\phi} = 1/2$ and

$$\begin{aligned} P_{B:2|\psi} &= \langle\psi| \hat{M}_2^\dagger \hat{M}_2 |\psi\rangle = \langle\psi| \frac{\hat{\mathbb{1}} + \boldsymbol{m}_2 \cdot \hat{\boldsymbol{\sigma}}}{3} |\psi\rangle = \frac{1 + \boldsymbol{m}_2 \cdot \boldsymbol{\psi}}{3} \\ &= \ = \frac{1 + \cos(\pi/3)}{3} = 1/2 \ . \end{aligned}$$

We similarly find $P_{B:1|\phi} = 1/2$. We find find that the probability of success is

$$P = 1/2 \ .$$

Hence, POVMs allow us to realize state discrimination with a better success probability than PVMs.