A Exercises

A.1 Bell inequalities

1. Consider the EPR state

$$|\mathrm{EPR}\rangle = \frac{|\!\uparrow\rangle_A \otimes |\!\downarrow\rangle_B - |\!\downarrow\rangle_A \otimes |\!\uparrow\rangle_B}{\sqrt{2}} \,. \tag{38}$$

   (a) We recall that the eigenstates of the spin operator along the axis oriented by the vector $\boldsymbol{u}$, characterized by the spherical angles θ and φ, read as

$$\begin{cases} |\uparrow_{\boldsymbol{u}}\rangle &= +\cos(\theta/2)\mathrm{e}^{-i\varphi/2}|\uparrow_z\rangle + \sin(\theta/2)\mathrm{e}^{+i\varphi/2}|\downarrow_z\rangle \\ |\downarrow_{\boldsymbol{u}}\rangle &= -\sin(\theta/2)\mathrm{e}^{-i\varphi/2}|\uparrow_z\rangle + \cos(\theta/2)\mathrm{e}^{+i\varphi/2}|\downarrow_z\rangle \end{cases}, \tag{39}$$

   see Ref. [26] for instance. Write the two-spin state $|\mathrm{EPR}\rangle$ in the basis $\{|\varepsilon_{\boldsymbol{a}}\rangle \otimes |\varepsilon_{\boldsymbol{b}}\rangle\}$ with $\varepsilon =\uparrow$ or $\downarrow$. For simplicity, assume $\varphi_A = \varphi_B$.
   Solution: Using the formulas for $|\uparrow_{\boldsymbol{u}}\rangle$ and $|\downarrow_{\boldsymbol{u}}\rangle$, we may write

$$\begin{cases} |\uparrow_z\rangle &= +\cos(\theta/2)\mathrm{e}^{+i\varphi/2}|\uparrow_{\boldsymbol{u}}\rangle - \sin(\theta/2)\mathrm{e}^{+i\varphi/2}|\downarrow_{\boldsymbol{u}}\rangle \\ |\downarrow_z\rangle &= +\sin(\theta/2)\mathrm{e}^{-i\varphi/2}|\uparrow_{\boldsymbol{u}}\rangle + \cos(\theta/2)\mathrm{e}^{-i\varphi/2}|\downarrow_{\boldsymbol{u}}\rangle \end{cases}.$$

   Then, using standard trigonometric relations, we find

$$\begin{aligned} |\mathrm{EPR}\rangle = \frac{1}{\sqrt{2}}\Bigg[&\sin\left(\frac{\theta_b-\theta_a}{2}\right)|\uparrow_{\boldsymbol{a}},\uparrow_{\boldsymbol{b}}\rangle + \cos\left(\frac{\theta_b-\theta_a}{2}\right)|\uparrow_{\boldsymbol{a}},\downarrow_{\boldsymbol{b}}\rangle \\ &-\cos\left(\frac{\theta_b-\theta_a}{2}\right)|\downarrow_{\boldsymbol{a}},\uparrow_{\boldsymbol{b}}\rangle + \sin\left(\frac{\theta_b-\theta_a}{2}\right)|\downarrow_{\boldsymbol{a}},\downarrow_{\boldsymbol{b}}\rangle\Bigg]. \end{aligned}$$

   (b) Conclude that the EPR state reads the same for any quantization axis, provided the same is used for both particles A and B, see Eq. (22).
   Solution: Using the previous equation for $\theta_A = \theta_B$, we immediately find

$$|\mathrm{EPR}\rangle = \frac{|\!\uparrow_{\boldsymbol{u}}\rangle_A \otimes |\!\downarrow_{\boldsymbol{u}}\rangle_B - |\!\downarrow_{\boldsymbol{u}}\rangle_A \otimes |\!\uparrow_{\boldsymbol{u}}\rangle_B}{\sqrt{2}} \,.$$

2. We aim at deriving the quantum average of the Bell correlator, Eq. (28). Consider a two-spin system in the EPR state (38) and assume simultaneous measurement of the spin components $\hat{\sigma}_{Aa} = \hat{\boldsymbol{\sigma}}_A \cdot \boldsymbol{a}$ and $\hat{\sigma}_{Bb} = \hat{\boldsymbol{\sigma}}_B \cdot \boldsymbol{b}$ are performed.

(a) What are the possible outcomes and with what probabilities are they found? You may use the results of exercise 1.

Solution: Here we use the notations $|+\rangle = |\uparrow\rangle$ and $|-\rangle = |\downarrow\rangle$. One can find $+$ or $-$ for both spins, with the joint probabilities

$$\begin{aligned} P_{++} &= \frac{1}{2}\sin^2\left(\frac{\theta_b-\theta_a}{2}\right) \\ P_{+-} &= \frac{1}{2}\cos^2\left(\frac{\theta_b-\theta_a}{2}\right) \\ P_{-+} &= \frac{1}{2}\cos^2\left(\frac{\theta_b-\theta_a}{2}\right) \\ P_{--} &= \frac{1}{2}\sin^2\left(\frac{\theta_b-\theta_a}{2}\right) \ . \end{aligned}$$

These formulas are readily found from the result of question 1(a).

(b) What are the possible results if we measure the spin of only one particle and with what probabilities are they found?

Solution: Particle A can be found in either $|+\rangle$ or $|-\rangle$, each with probability $P_\pm = P_{\pm+} + P_{\pm-}$. It yields

$$P_\pm = 1/2.$$

The same result is found for particle B.

(c) What is the conditional probability that the measurement on particle B gives the result $+$, assuming that the measurement on particle A has given $-$? Recover this result by applying the wave packet collapse rule.

Solution: We can write

$$P_{B:+|A:-} = P_{-+}/P_- \ ,$$

so that

$$P_{B:+|A:-} = \cos^2\left(\frac{\theta_b-\theta_a}{2}\right) \qquad \text{and} \qquad P_{B:-|A:-} = \sin^2\left(\frac{\theta_b-\theta_a}{2}\right) \ .$$

One can recover these results by performing a measurement of the spin state of particle A. If say $-$ is found, project the two-particle state on $|-\boldsymbol{a}\rangle_A$, renormalize the projected state, and compute the square of the amplitude of $|A:-\boldsymbol{a}, B:\pm\boldsymbol{b}\rangle$. It yields

$$|\psi_{|A:-\boldsymbol{a}}\rangle_B = -\cos\left(\frac{\theta_b-\theta_a}{2}\right)|+\boldsymbol{b}\rangle + \sin\left(\frac{\theta_b-\theta_a}{2}\right)|-\boldsymbol{b}\rangle \ ,$$

and we recover the previous results.

(d) It is assumed (only here) that $\boldsymbol{a} = \boldsymbol{b}$. Show that the result of the measurement of one spin is perfectly determined by that of the other spin. What can be said about the correlation between the results of these measurements?

Solution: For $\boldsymbol{a} = \boldsymbol{b}$, i.e. $\theta_a = \theta_b$, one finds

$$P_{B:+|A:-} = 1 \qquad \text{and} \qquad P_{B:-|A:-} = 0 \ .$$

The results of the two measurements are thus perfectly anti-correlated.

(e) Show that the quantity $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = \langle \hat{\sigma}_{1\boldsymbol{a}} \hat{\sigma}_{2\boldsymbol{b}} \rangle$ may be written as $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b}$. You may use the probabilities found above.

Solution: The quantity $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b})$ is the average product of the measurement results on each spin. One thus finds

$$E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = P_{++} + P_{--} - P_{+-} - P_{-+} = -\cos\left(\theta_B - \theta_A\right) \ ,$$

that is

$$E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b} \ .$$

(f) The vector $\boldsymbol{a}$ being fixed, we consider the situation where the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}'$, and $\boldsymbol{b}'$ form the same angle θ with the previous vector, see Fig. 2(b). Find the expression of the Bell correlation function

$$\langle C \rangle_{\rm Q} = E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) - E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}') + E_{\rm Q}(\boldsymbol{a}', \boldsymbol{b}) + E_{\rm Q}(\boldsymbol{a}', \boldsymbol{b}') \tag{40}$$

as a function of the angle θ.

Solution: For this choice of angles between the vectors, one finds $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = E_{\rm Q}(\boldsymbol{a}', \boldsymbol{b}) = E_{\rm Q}(\boldsymbol{a}', \boldsymbol{b}') = -\cos(\theta)$ and $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}') = -\cos(3\theta)$. Hence we find

$$\langle C \rangle_{\rm Q} = \cos(3\theta) - 3\cos(\theta) \ .$$

(g) Find the extrema of the correlation function $\langle C \rangle_{\rm Q}$ versus θ and show that there is a conflict between quantum theory and local hidden variable theories.

Solution: The correlation function $\langle C \rangle_{\rm Q}$ shows extrema at $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$, ... and points at

$$\max\left(|\langle C \rangle_{\rm Q}|\right) = 2\sqrt{2} > 2 \ .$$

It contrast, local hidden variable (LHV) theories predict $|\langle C \rangle_{\rm LHV}| \leq 2$, see Eq. (26) and discussion of Sec. 2.2.

3. Show that for any product state of the spins A and B, the average value of the quantum Bell correlator, Eq. (27), fulfills the classical Bell inequality, $|\langle C \rangle_{\rm Q}| \leq 2$.

You may use $\langle\psi|\,\hat{\boldsymbol{\sigma}}\,|\psi\rangle = \boldsymbol{\psi}$, where $\hat{\boldsymbol{\sigma}}$ is the vector spin operator and $\boldsymbol{\psi}$ is the Bloch vector associated to the ket $|\psi\rangle$.

Solution: For a product state $|\Psi\rangle = |\psi\rangle_A \otimes |\chi\rangle_B$ and $\hat{E}(\boldsymbol{a},\boldsymbol{b}) = \big(\hat{\boldsymbol{\sigma}}_A\cdot\boldsymbol{u}\big)\otimes\big(\hat{\boldsymbol{\sigma}}_B\cdot\boldsymbol{v}\big)$, we have

$$\langle\Psi|\,\hat{E}(\boldsymbol{a},\boldsymbol{b})\,|\Psi\rangle = \sigma_A \times \sigma_B \ ,$$

with $\sigma_A = \boldsymbol{a}\cdot\boldsymbol{\psi}$ and $\sigma_B = \boldsymbol{b}\cdot\boldsymbol{\chi}$. Using similar notations for $\hat{\sigma}'_A$ and $\hat{\sigma}'_B$, we then find

$$\langle C\rangle_{\mathrm{Q}} = \sigma_A\big(\sigma_B - \sigma'_B\big) + \sigma'_A\big(\sigma_B + \sigma'_B\big)$$

and, since $|\sigma_A|, |\sigma'_A| \leq 1$,

$$|\langle C\rangle_{\mathrm{Q}}| = |\sigma_B - \sigma'_B| + |\sigma_B + \sigma'_B| \ .$$

Since σ_B and σ'_B play symmetric roles, we may assume $|\sigma_B| \geq |\sigma'_B|$, and $\sigma_B \geq 0$ (otherwise, we may substitute σ_B for $-\sigma_B$, which does not change the formula). We then find

$$|\langle C\rangle_{\mathrm{Q}}| = \sigma_B - \sigma'_B + \sigma_B + \sigma'_B = 2\sigma_B \leq 2 \ .$$

This is nothing but the Bell inequality valid of any LHV theory, Eq. (26).

A.2 Quantum teleportation

4. Starting from Eq. (35), prove Eq. (36).

   Solution: Starting from the initial 3-qubit state (35), we may write

$$\begin{aligned}
|\Psi\rangle_{AMB} &= \frac{\alpha}{\sqrt{2}}|001\rangle_{AMB} - \frac{\alpha}{\sqrt{2}}|010\rangle_{AMB} + \frac{\beta}{\sqrt{2}}|101\rangle_{AMB} - \frac{\beta}{\sqrt{2}}|110\rangle_{AMB} \\
&= \frac{\alpha}{\sqrt{2}}|\underbrace{00}_{\frac{|\mathcal{B}_{00}\rangle+|\mathcal{B}_{10}\rangle}{\sqrt{2}}}1\rangle - \frac{\alpha}{\sqrt{2}}|\underbrace{01}_{\frac{|\mathcal{B}_{01}\rangle+|\mathcal{B}_{11}\rangle}{\sqrt{2}}}0\rangle + \frac{\beta}{\sqrt{2}}|\underbrace{10}_{\frac{|\mathcal{B}_{01}\rangle-|\mathcal{B}_{11}\rangle}{\sqrt{2}}}1\rangle \\
&\quad -\frac{\beta}{\sqrt{2}}|\underbrace{11}_{\frac{|\mathcal{B}_{00}\rangle-|\mathcal{B}_{10}\rangle}{\sqrt{2}}}0\rangle \\
&= \frac{1}{2}\Big[|\mathcal{B}_{00}\rangle_{AM}\otimes\big(+\alpha|1\rangle - \beta|0\rangle\big)_B + |\mathcal{B}_{01}\rangle_{AM}\otimes\big(-\alpha|0\rangle + \beta|1\rangle\big)_B \\
&\quad + |\mathcal{B}_{10}\rangle_{AM}\otimes\big(+\alpha|1\rangle + \beta|0\rangle\big)_B + |\mathcal{B}_{11}\rangle_{AM}\otimes\big(-\alpha|0\rangle - \beta|1\rangle\big)_B\Big] \ ,
\end{aligned}$$

   which is Eq. (36).