Homework 3 - Entanglement

1.1 1. $\vec{u} = \langle \vec{\sigma} \rangle = Tr(\rho \vec{\sigma}) = \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix}$,

with $u_x = \langle \sigma_x \rangle = Tr[\rho \sigma_x]$

$$= Tr\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \rho_{01} + \rho_{10} = 2 Re\, \rho_{01}$$

$$u_y = \langle \sigma_y \rangle = Tr\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = i(\rho_{01} - \rho_{10}) = 2 Im\, \rho_{10}$$

$$u_z = \langle \sigma_z \rangle = \rho_{00} - \rho_{11}$$

2. ρ must have positive eigenvalues, given by the equation:

$$\begin{vmatrix} \rho_{00} - \lambda & \rho_{01} \\ \rho_{10} & \rho_{11} - \lambda \end{vmatrix} = \lambda^2 - \lambda \cancel{\ } |\rho_{10}|^2 + \rho_{00}\rho_{11} = 0$$

The product of $\lambda_1 \lambda_2 \geq 0 \Rightarrow \rho_{00}\rho_{11} \geq |\rho_{10}|^2$ ~~as~~ $\rho_{00}\rho_{11} - |\rho_{10}|^2 = \lambda_1 \lambda_2$

3. $$\|\vec{u}\|^2 = 4\left[(Re\, \rho_{01})^2 + (Im\, \rho_{01})^2 \right] + (\rho_{00} - \rho_{11})^2$$

$$= 4 |\rho_{01}|^2 + (\rho_{00} - \rho_{11})^2 \leq 4 \rho_{00}\rho_{11} + (\rho_{00} - \rho_{11})^2 = (\rho_{00} + \rho_{11})^2$$

As $Tr\rho = 1 \Rightarrow |u|^2 \leq 1$

1.2 1. - $\rho_{\pm} = \frac{1}{4} (|00\rangle + |01\rangle + |10\rangle \pm |11\rangle)(\langle 00| + \langle 01| + \langle 10| \pm \langle 11|)$

- $\rho_+^A = {}_B\langle 0 | \rho_\pm | 0 \rangle_B + {}_B\langle 1 | \rho_\pm | 1 \rangle_B = Tr_B \rho$ (or $Tr_A \rho$)

$$= \frac{1}{4} (|0\rangle + |1\rangle)(\langle 0| + \langle 1|) + \frac{1}{4} (|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ possesses diagonal terms $\Rightarrow \varphi_+$ non entangled.

(for an entangled state, you expect a diagonal reduced density matrix)

- $\rho_-^A = \frac{1}{4} (|0\rangle + |1\rangle)(\langle 0| + \langle 1|) + \frac{1}{4} (|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$

$$= \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|)$$ The matrix is mixed (even maximally) $\Rightarrow |\varphi_-\rangle$ is entangled.

2_ $$|\psi\rangle\langle\psi| = \rho = \frac{1}{3}\left(|00\rangle + |01\rangle + |11\rangle\right)\left(\langle 00| + \langle 01| + \langle 11|\right)$$

$$\mathrm{Tr}_B \rho = \rho_A = {}_B\langle 0|\rho|0\rangle_B + {}_B\langle 1|\rho|1\rangle_B$$
$$= \frac{1}{3}|0\rangle_A\langle 0|_A + \frac{1}{3}\left(|0\rangle + |1\rangle\right)_A\left(\langle 0| + \langle 1|\right)_A$$
$$= \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

3_ $$\rho = \sum_{i=0,xyz}\sum_{j=0,xyz}\lambda_{ij}\,\sigma_i\otimes\sigma_j$$

$$\langle\sigma_i\otimes\sigma_j\rangle = \mathrm{Tr}\left[\rho\,\sigma_i\otimes\sigma_j\right] = \mathrm{Tr}\sum_k\sum_l \lambda_{kl}\sigma_k\otimes\sigma_l\;\sigma_i\otimes\sigma_j$$
$$= \mathrm{Tr}\sum_{kl}\lambda_{kl}\;\sigma_k\sigma_i\otimes\sigma_l\sigma_j$$
$$= \sum_{kl}\lambda_{kl}\;\underbrace{\mathrm{Tr}\left[(\sigma_k\sigma_i)\otimes(\sigma_l\sigma_j)\right]}_{\left[\underbrace{\mathrm{Tr}(\sigma_k\sigma_i)}_{2\delta_{ki}}\right]\left[\underbrace{\mathrm{Tr}(\sigma_l\sigma_j)}_{2\delta_{lj}}\right]}$$
$$= 4\lambda_{ij}$$

4_ (34) $${}_1\langle 0|\rho_N^{GHZ}|0\rangle_1 + {}_1\langle 1|\rho_N^{GHZ}|1\rangle_1$$

The particle we trace over

$$= \frac{1}{2}\left(|0\rangle^{\otimes N-1}\langle 0|^{\otimes N-1} + |1\rangle^{\otimes N-1}\langle 1|^{\otimes N-1}\right)$$

If you are ensure, take $\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

$$\rho_3 = \frac{1}{2}\left(|000\rangle + |111\rangle\right)\left(\langle 000| + \langle 111|\right)$$

Then $${}_1\langle 0|\rho_3|0\rangle_1 + {}_1\langle 1|\rho_3|1\rangle_1 = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

(35) $$\rho_N = |W_N\rangle\langle W_N| \longrightarrow {}_1\langle 0|W_N\rangle\langle W_N|0\rangle_1 + {}_1\langle 1|W_N\rangle\langle W_N|1\rangle_1$$
$$= \frac{1}{N}\left(\underbrace{|100..\rangle + |01...\rangle}_{N-1} + \cdots\right)\left(\underbrace{\langle 100\cdots| + \langle 010\cdots| + \cdots}_{N-1}\right)$$
$$+ \frac{1}{N}\;|\underbrace{00..0}_{N-1}\rangle\langle\underbrace{0...0}_{N-1}|$$

Hence $\rho_N = \frac{\sqrt{N-1}\sqrt{N-1}}{N} |W_{N-1}\rangle\langle W_{N-1}| + \frac{1}{N} |0 0 \cdots 0\rangle\langle 0 \cdots 0|$

2- Entropies

1. $S_{VN}(\rho_A) = -\mathrm{Tr}[\rho_A \log \rho_A] = \sum_{n=1}^{r} \langle n|\rho_A \log \rho_A|n\rangle$, where $\{|n\rangle\}$ is the eigenbasis of ρ_A associated to the eigenvalues λ_n.
(use e.g. the Schmidt decomposition $\rho_A = \sum_{n=1}^{r} \lambda_n |n\rangle\langle n|$)

Then $\rho \log \rho$ is diagonal in this basis

and $S_{VN} = -\sum_{n=1}^{r} \lambda_n \log \lambda_n$

2. If $\rho = \sum_k p_k |k\rangle\langle k|$, then $S_R^{(\alpha)}(\rho) = \frac{1}{1-\alpha} \log[\mathrm{Tr}\rho^\alpha]$

$$S_R(\rho) = \frac{1}{1-\alpha} \log \sum_k p_k^\alpha$$

3. $\sum_k p_k^\alpha = \sum_k p_k \, p_k^{\alpha-1} = \sum_k p_k \, e^{(\alpha-1)\log p_k}$.

if $\alpha \to 1$, we Taylor expand the exponential:

$$\sum p_k^\alpha \simeq 1 + (\alpha-1) \sum_k p_k \log p_k.$$

as $e^{\alpha-1 \log p_k} = 1 + (\alpha-1)\log p_k$ and $\sum p_k = 1$

Thus: $S_R^{(\alpha)} \simeq \frac{1}{1-\alpha} \log\left[1 + (\alpha-1)\sum_k p_k \log p_k\right]$

$\simeq -\sum p_k \log p_k$ when $\alpha \to 1$

$= S_{VN}$

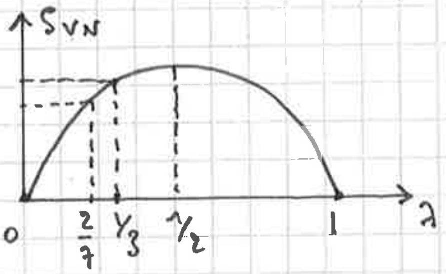
4- Let us calculate the reduced ρ:

$\rho_1^A = \frac{2}{7}|0\rangle\langle 0| + \frac{5}{7}|1\rangle\langle 1|$, while $\rho_2^A = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$

$S_{VN}(\rho_A^1) = -\frac{2}{7}\log\frac{2}{7} - \frac{5}{7}\log\frac{5}{7} = 0{,}598$

$S_{VN}(\rho_2^A) = -\frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{1}{3} = 0{,}636.$

graphically



$S_{VN} = -\lambda \log \lambda - (1-\lambda)\log(1-\lambda)$

$S_{VN}^{(1)} < S_{VN}^{(2)} \Rightarrow |\varphi_2\rangle$ is more entangled with this criteria.

5- $S_1^{(2)} = + \frac{1}{1-2} \log\left[\left(\frac{2}{7}\right)^2 + \left(\frac{5}{7}\right)^2\right] = 0,524$

$S_2^{(2)} = \frac{1}{1-2} \log\left[\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right] = 0,588$

Same conclusion