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# Lecture 3: Quantum entanglement I

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As already mentioned in the first lecture, entanglement is a consequence of the superposition principle applied to several objects or several degrees of freedom of the same object. For example, two spins 1/2, $A$ and $B$ may be prepared in a *classical* configuration where the first spin is in $|\uparrow\rangle$ and the second in $|\downarrow\rangle$, leading to the two spin state $|\uparrow\rangle_A \otimes |\downarrow\rangle_B = |\uparrow_A, \downarrow_B\rangle$. The state $|\downarrow_A, \uparrow_B\rangle$ is also a possibility. The superposition principle allows one to place them in the state:

$$|+\rangle_{AB} = \frac{1}{2}(|\uparrow_A, \downarrow_B\rangle + |\downarrow_A, \uparrow_B\rangle) \ . \tag{1}$$

So far, no new ideas. What is now surprising is that in such a two-spin state, one can not assign a quantum state to any of the two spins, *i.e.*, one cannot find two states $|\psi_A\rangle$ and $|\psi_B\rangle$ such that $|+\rangle_{AB} = |\psi_A\rangle \otimes |\psi_B\rangle$. States for which such a decomposition exists are called *separable* or *product states*. Product states corresponds to our intuitive vision of the world that one can assign a property to a subpart of a system independently of the other parts. This is called *local realism*. The ones for which the decomposition is impossible are called *entangled* for reasons that will become clearer in the lecture: for such states local realism breaks down...

Entanglement is in a sense not a new concept: it is rather the consequence of extending the superposition principle, accurately tested at the single particle level, to the case of two or more particles. But the properties of such states are very surprising as they imply the existence of quantum correlations stronger than any possible classical correlations between the two objects. Moreover, while superpositions are ubiquitous in wave physics and have therefore many counterparts in classical physics, entanglement is a genuine quantum feature, with no classical counterpart. This was pointed out already in the 1920s, in particular by Schrödinger. Since then, it was gradually realized that entanglement is not just a curiosity of the quantum world, but plays a role in many physical phenomena and can be used as a resource for quantum information tasks.

In this lecture, we will give an overview of the properties of entangled states, discuss a few examples and methods to generate them. We will also discuss the foundational aspects and in particular how the violation of Bell's inequalities modifies the way we must understand physical reality. The next lecture will deal with the still largely open question of the characterization of entanglement.

## 1 Properties of entangled states

We consider here two particles $A$ and $B$, or two degrees of freedom such as position or spin of an electron. Each is described by a Hilbert space $\mathcal{E}_{A,B}$ with basis $\{|a\rangle\}$ and

$\{|b\rangle\}$. The system $A+B$ is then characterized by the tensor product of the two spaces $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$ and any states can be decomposed on the basis obtained by forming the tensor product of the basis of each part:

$$|\psi\rangle_{AB} = \sum_{a=1}^{\dim\mathcal{E}_A} \sum_{b=1}^{\dim\mathcal{E}_B} c_{ab} |a, b\rangle \ . \tag{2}$$

As already mentioned, the states that can be factorized in $|\psi\rangle_{AB} = |\psi_A\rangle \otimes |\psi_B\rangle$ are called *separable states*, or *product states*. All the others are named *entangled*. This definition can be extended to larger numbers of particles or degrees of freedom.

Before going further, let us first note that product states are exceptions. To understand this, consider a system of $N$ particles, each described by a Hilbert space of dimension $d$. An arbitrary state can we written:

$$|\Psi\rangle = \sum_{n_1,\ldots,n_N} c_{n_1,\ldots,n_N} |n_1\rangle \otimes \ldots \otimes |n_N\rangle \ , \tag{3}$$

and requires $d^N$ independent coefficients (up to the normalization and global phase), which is also the dimension of the Hilbert state of the whole system. A product state describing the $N$ particles reads:

$$|\Psi\rangle_{\text{prod}} = \Big( \sum_{n_1=1}^{d} c_{n_1} |n_1\rangle \Big) \otimes \ldots \otimes \Big( \sum_{n_N=1}^{d} c_{n_N} |n_N\rangle \Big) \ , \tag{4}$$

and only $N \times d$ independent coefficients are necessary (again up to the normalization and global phase). For $N \gg 1$, there are thus exponentially more entangled states than product states.

Let us now consider the specific case of two quantum bits to illustrate the properties of entangled states. The following four entangled states, called *Bell states*, form a basis of the two-qubit Hilbert space:

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \tag{5}$$

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \ . \tag{6}$$

To check that these are indeed entangled states of the two qubits, one simply shows that a decomposition of the form $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$ is not possible.

The first important property is the fact that any *local rotation* of the state of one qubit can not "dis-entangle" the state. Consider for example the rotation $|0\rangle \to (|0\rangle - i|1\rangle)/\sqrt{2}$ and $|1\rangle \to (-i|0\rangle + |1\rangle)/\sqrt{2}$ (see Eq. (18) of Lecture 1). Then $|\psi_+\rangle \to -i(|01\rangle + |10\rangle)/\sqrt{2}$, which is still entangled. This also means that to generate entanglement, one must use an operation that couples the two sub-parts, as we will see in the next section.

The second important property is the fact that the states feature *strong correlations between their subparts*, despite the fact that a measurement on a subpart gives a random outcome. Consider once again the state $|\psi_+\rangle$. The probability to measure the first qubit in $|0\rangle$ is $p(A:0)=1/2$, *i.e.* the outcome is random. Similarly, $p(A:1)=p(B:0)=p(B:1)=1/2$. However the probability to obtain 0 on the second qubit when you have measured 0 on the first one is the conditional probability $p(B:0|A:0)=p(A:0\&B:0)/p(A:0)=(1/2)/(1/2)=1$. The correlation between the result is perfect, even if they are so distant that the measurement of one subpart of the system can not influence the outcome of a measurement on the other part. The correlations are therefore very strong, and non-local. This may look weird, but this is not unknown to the classical world: two classical objects possessing a given property can be correlated. Take for example two balls, a red and a blue. If I hand you the red one, you know for sure that I have the blue. As we will see when discussing the violation of Bell's inequalities, the quantum correlations are however stronger than the classical ones.

## 2 Examples of entangled states and preparation

As seen previously, a local operation on a product state cannot generate entanglement. This means that one must consider operations that act on the different parts of a subsystems. This operations can either be an interaction between the various parts, or a measurement performed on a the system which does not distinguish between the parts.

**Entanglement by interaction.** Almost any interaction or coupling between a system $A$ and a system $B$ (or more generally two degrees of freedom of a quantum system) produces an entangled state. The two subsystems may then separate spatially, thus no longer interact, while preserving the entanglement.

As a first example, let us come back to the Jaynes-Cumming Hamiltonian introduced in the second lecture. The interaction between a two-level atom an a monomode cavity field is described by the Hamitonian (near-resonant approximation):

$$H_{\mathrm{JC}}=\frac{\hbar\Omega}{2}(\hat{\sigma}^{+}\hat{a}++\hat{\sigma}^{-}\hat{a}^{+})\ . \tag{7}$$

Preparing the atom in $|1\rangle$ in a initially empty cavity, the state of the (atom+field), $|\psi(0)\rangle=|1,n=0\rangle$, evolves into $|\psi(t)\rangle=\cos\frac{\Omega t}{2}|1,n=0\rangle-i\sin\frac{\Omega t}{2}|0,n=1\rangle$. Choosing $\Omega t=\pi/2$, we obtain the entangled state $(|1,n=0\rangle-i\,|0,n=1\rangle)/\sqrt{2}$. This entanglement between atom and field can also lead to an entanglement between two atoms. Assume that you send a first atom $A$ through the cavity such that its interaction with the cavity field is $\Omega t=\pi/2$; after it has left the cavity, $|\psi_{A,\mathrm{field}}\rangle=(|1_A,n=0\rangle-i\,|0_A,n=1\rangle)/\sqrt{2}$. If you send as second atom $B$, initially in state $|0\rangle$ through the cavity so that $\Omega t=\pi$, the states of the (cavity+atoms) is $|\psi_{AB,\mathrm{field}}\rangle=(|1_A,0_B\rangle-i\,|0_A,1_B\rangle)/\sqrt{2}\otimes|n=0\rangle$. The two atoms are now entangled.

As a second example, consider the Stern and Gerlach experiment, where entanglement appears between two degrees of freedom of a single particle. In the experiment, silver atoms (carrying a spin 1/2) are emitted from an oven in an arbitrary spin state. Assuming that an atom is emitted with a well-defined momentum, it is initially in the momentum-spin product state $|\boldsymbol{p}_0\rangle\,(c_+\,|\uparrow\rangle+c_-\,|\downarrow\rangle)$. The atom then passes through a magnetic field gradient $\boldsymbol{B}(\boldsymbol{r})$ and is deflected up or down depending on its spin state.

The evolution of the atom is governed by the Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} - \hat{\boldsymbol{\mu}} \cdot \boldsymbol{B}(\hat{\boldsymbol{r}}), \tag{8}$$

where $\hat{r}$ is the position and $\hat{p}$ the momentum of the atom, $m$ its mass, and $\hat{\boldsymbol{\mu}}$ its magnetic moment. Assuming that the wavepacket is sufficiently localized, with the initial momentum $\boldsymbol{p}_0$, and that the magnetic field has a gradient along, say the $z$ direction, the state at time $t$ is given by

$$|\psi(t)\rangle = c_+ |\boldsymbol{p}_0 + \Delta\boldsymbol{p}\rangle \otimes |\uparrow\rangle + c_- |\boldsymbol{p} - \Delta\boldsymbol{p}\rangle \otimes |\downarrow\rangle \,, \tag{9}$$

where $\Delta\boldsymbol{p}$ is determined by the classical trajectory (Ehrenfest theorem). Hence, while the atom was initially in a motion-spin product state, the interaction with the magnetic field entangles the two degrees of freedom. The reason is that the effect of the magnetic field on the (semi-classical) trajectory depends on the spin state. This produces the coupling between the degrees of freedom at the origin of their entanglement.

As a third example, let us look at the action of a 50/50 beamsplitter on a single photon. The beam splitter has two input modes $a, b$ and two output modes $c, d$. It mixes the two modes according to the unitary transformation:

$$\hat{E}_c^- = \frac{1}{\sqrt{2}}(\hat{E}_a^- + \hat{E}_b^-) \tag{10}$$

$$\hat{E}_d^- = \frac{1}{\sqrt{2}}(\hat{E}_a^- - \hat{E}_b^-) \,, \tag{11}$$

where $\hat{E}_a^- = \mathcal{E}_\mathrm{v}\hat{a}^+$ is the field operator ($\mathcal{E}_\mathrm{v}$ is the vacuum field amplitude), related to the creation operators. Hence, a single photon state in mode $a$ is $|1_a, 0_b\rangle = \hat{a}^+ |0\rangle = (\hat{c}^+ + \hat{d}^+)/\sqrt{2}\,|0\rangle = (|1_c, 0_d\rangle + |0_c, 1d\rangle)/\sqrt{2}$: the two modes $c$ and $d$ are entangled! This example is quite subtle, as what is entangled are now the modes and not the particle (there is only one...). Here, entanglement comes from the fact that the beamsplitter acts on the two modes.

As a final example, consider the spontaneous emission of a photon by an atom initially excited. It results from the interaction of the atom with the quantized radiation field, as you have learned in quantum optics. For a two-level atom (ground state $|g\rangle$ and excited state $|e\rangle$), the interaction Hamiltonian is

$$\hat{H}_\mathrm{int} = \sum_\ell \left( \hbar g_\ell |e\rangle \langle g| \hat{a}_\ell + \hbar g_\ell^* |g\rangle \langle e| \hat{a}_\ell^\dagger \right) , \tag{12}$$

where $\hat{a}_\ell$ and $\hat{a}_\ell^\dagger$ are the annihilation and creation operators of a photon in the electromagnetic mode $\ell$ and $g_\ell$ is the mode-dependent coupling constant. Assuming that the atom is initially in the excited state and the radiation field in the vacuum, $|\Psi(0)\rangle = |e\rangle \otimes |0\rangle$, the interaction between the two yields

$$|\Psi(0)\rangle = c_0(t) |e\rangle \otimes |0\rangle + \sum_\ell c_\ell(t) |g\rangle \otimes |1_\ell\rangle \,, \tag{13}$$

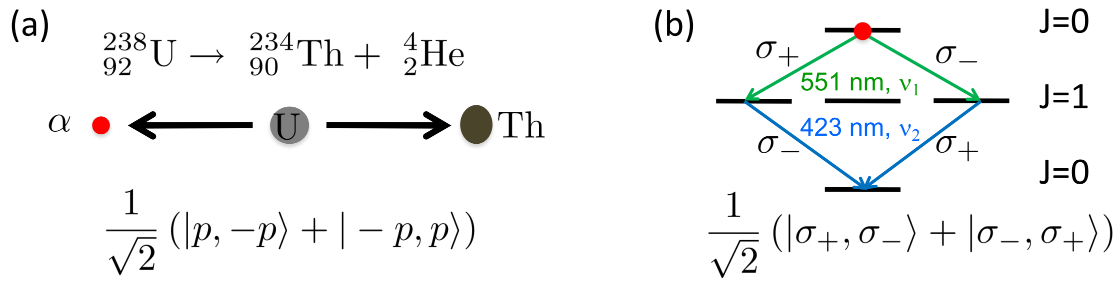


Figure 1: *(a) The radioactive $\alpha$-decay of a uranium atom leads to two particles with entangled momentum state. (b) The radiative decay of the $^{40}$Ca atom leads to two photons with different frequencies and entangled polarizations.*

where $|1_\ell\rangle$ is the state of the radiation field with one photon in the mode $\ell$. The state is entangled, with the atom partially in the ground state and partially in the excited state, and the radiation field partially filled by a photon (in either mode $\ell$).

**Entanglement as a consequence of conservation laws.** It is not always easy to write the interaction Hamiltonian between several parts of a subsystem, often because we don't know the details of the interaction. However, based on general considerations, we can infer that entanglement is generated.

Consider for example the decay of a particle $a$ at rest into two particles $b$ and $c$ (see Fig. 1a). The conservation of momentum implies that one product particle has momentum $p$, while the other has momentum $-p$. Hence the state of the two products is $(|p,-p\rangle + |-p,p\rangle)/\sqrt{2}$, which is entangled.

Similarly, consider the atomic cascade in an atom such as calcium (see Fig. 1b). When placed in state $|e\rangle$, the atom can decay to the ground state $|g\rangle$ by emitting two successive photons at different frequencies $\nu_1$ and $\nu_2$ along two decay channels: either the first photon is emitted with a polarization $\sigma^+$, the second having a polarization $\sigma^-$ or vice versa. This polarization is a consequence of the conservation of angular momentum of the (atom + photon) system in a radiative decay. Hence the state of the photons after the atoms has decayed is $(|\sigma^+_{\nu_1},\sigma^-_{\nu_2}\rangle + |\sigma^-_{\nu_1},\sigma^+_{\nu_2}\rangle)/\sqrt{2}$. Introducing the horizontal and vertical polarizations $h, v$ and the fact that $\sigma^\pm = \mp(h \pm iv)$, the photon state is $(|hh\rangle + |vv\rangle)/\sqrt{2}$.

**Conditional preparation by a measurement.** It may actually be that the parts of the system never interact but that a measurement on the ensemble generates an entangled state. Consider for instance $N$ two-level atoms with states $|0\rangle, |1\rangle$ coupled by a microwave transition (as in the hyperfine structure of Rb). Drive the atoms with a weak microwave field that prepares each atom in $|0\rangle + \epsilon\,|1\rangle$, with $\epsilon \ll 1$. The $N$-atom state is thus

$$|\psi_N\rangle \sim (|0\rangle + \epsilon\,|1\rangle)^{\otimes N} = |0,0,0,...\rangle + \epsilon(|1,0,0,...\rangle + |0,1,0,...\rangle + ... + |0,0,...,1\rangle) + \mathcal{O}(\epsilon^2)\,. \tag{14}$$

If one now measures (in a non-destructive way, see next lecture) the state of the atoms and finds that at least one atom is in $|1\rangle$ (this occurs with a probability $N\epsilon^2$), the state after the measurement is

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{N}}(|1,0,0,...\rangle + |0,1,0,...\rangle + ... + |0,0,...,1\rangle) \ , \tag{15}$$

with the other contributions of order $\epsilon$, *i.e.* negligible.

**Particles can be entangled in different ways.** Most examples above used two particles. As soon as more than 3 particles are involved it turns out that there are different ways to be entangled. Consider the two following 3-qubit states

$$|\mathrm{W}\rangle \ = \ \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle) \tag{16}$$

$$|\mathrm{GHZ}\rangle \ = \ \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle) \ . \tag{17}$$

Here “W” stands for Werner, and “GHZ” for Greebergen, Horne, Zeilinger the names of the physicists who invented them. If you apply a local rotations on those two states, you will never be able to transform the state $|\mathrm{W}\rangle$ into $|\mathrm{GHZ}\rangle$. This is different from the two qubit case, where you can transform any Bell state into another one by local rotations. One then introduces *entanglement classes*, which are ensembles of states that can be transformed one into another by local operations. For instance, $|\mathrm{GHZ}\rangle$ and $(|0,1,0\rangle + |1,0,1\rangle)/\sqrt{2}$ are different states, but belong to the same classe. For three qubits, there exits just 2 classes (W and GHZ). For four qubits, there are 9 classes, and for more qubits we don't know...

# 3 Entanglement is fragile...

Entanglement is generated by almost any interaction between two systems, and this is what makes it very fragile. Since a quantum system can never be perfectly isolated, it necessarily interacts with its environment. The system's state then gets rapidly entangled with that of the environment, which destroys the quantum coherences of the system. Think about the example of spontaneous emission seen in the previous section: there the electromagnetic field acts like the environment that gets entangled with the atom. This process is called *decoherence* and will be discussed in lecture 8. It is all the more rapid that the system is large: if a single atom undergoes decoherence in a characteristic time $\tau$, an ensemble of $N$ atoms loses its coherence in a time $\sim \tau/N$. We will see in lectures 6 and 7 that the time $\tau$ is related to coupling strength $\kappa$ between the system and the environment, $\tau \sim 1/\kappa$.

Here we will rather consider the role of “classical noise” on various entangled states. Consider the $N$-qubit GHZ state,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle). \tag{18}$$

All particles are entangled with each other. Assume now that among all these particles, we measure the state of the last one in the computational basis $\{|0\rangle, |1\rangle\}$. If we measure it in $|0\rangle$, then all particles are projected onto $|0\rangle$, i.e. $|00...0\rangle$ with probability 1/2, while measuring it in $|1\rangle$ projects onto $|11...1\rangle$, again with probability 1/2. In both cases, we get a product state and entanglement is totally destroyed. However, if we measure in another basis, the entanglement may be preserved. For instance, introducing the basis $\{|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}\}$,

$$|\mathrm{GHZ}\rangle_N = \frac{|00...0\rangle_{N-1} + |11...1\rangle_{N-1}}{\sqrt{2}} \otimes |+\rangle + \frac{|00...0\rangle_{N-1} - |11...1\rangle_{N-1}}{\sqrt{2}} \otimes |-\rangle . \quad (19)$$

Measuring in the basis $\{|+\rangle, |-\rangle\}$, we get $(|00...0\rangle_{N-1} + |11...1\rangle_{N-1})/\sqrt{2}$ with probability 1/2, and $(|00...0\rangle_{N-1} - |11...1\rangle_{N-1})/\sqrt{2}$ with probability 1/2. In this case, entanglement is preserved, but the state is strongly affected in one out of two cases.

There exists however states that are better protected against decoherence induced by the measurement. Consider for instance a measurement in the computational basis applied to the first qubit of a $N$-qubit W state. If we obtain 0, the state after the measurement is

$$|\psi'\rangle \propto |0\rangle\langle 0| \otimes \hat{\mathrm{Id}}_{N-1} |\mathrm{W}_N\rangle = \frac{1}{\sqrt{N-1}}(|100...00\rangle + |010...00\rangle + ... + |000...01\rangle) , \quad (20)$$

with probability $P_0 = 1 - 1/N$. If the result is 1, the state after the measurement is $|000...00\rangle$ with probability $P_1 = 1/N$. Hence, in almost all cases (for $N \gg 1$), we get a $(N-1)$-qubit W state, and entanglement is preserved.

As a final example, let us consider a $N$-atom GHZ state, with the two qubit states separated by a transition at a frequency $\omega$. As time evolves, the states becomes:

$$|\mathrm{GHZ}(t)\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + e^{-iN\omega t} |11...1\rangle). \quad (21)$$

Suppose now that, due to fluctuations of the environment, the frequency varies according to a Gaussian statistics around a value $\omega_0$: $p(\omega) = \exp\left[-(\omega-\omega_0)^2/(2\Delta\omega^2)\right]/(\sqrt{2\pi}\Delta\omega)$. Then the average coherence between the two parts of the superposition is: $\langle e^{-i\omega t}\rangle = \exp\left[-N^2\Delta\omega^2 t^2\right]\exp[-i\omega_0 t]$. Hence, after a time $\sim t/(N\Delta\omega)$ the coherence of the state is lost and it behaves as a statistical mixture of $|00...0\rangle$ and $|11...1\rangle$. This example is also an illustration that the sensitivity of entangled states to fluctuations increases with their size.

# 4 Quantum correlations and Bell's inequalities

Let us now come back to the existence of non-local correlations between the subparts of an entangled system and try to understand its implication. As we will see, they put strong constraints on our understanding of physical reality.

**The Einstein, Podolsky, and Rosen argument.** As is well-known, Einstein was not happy with the, apparently fundamental, probabilistic aspect of quantum physics ('God

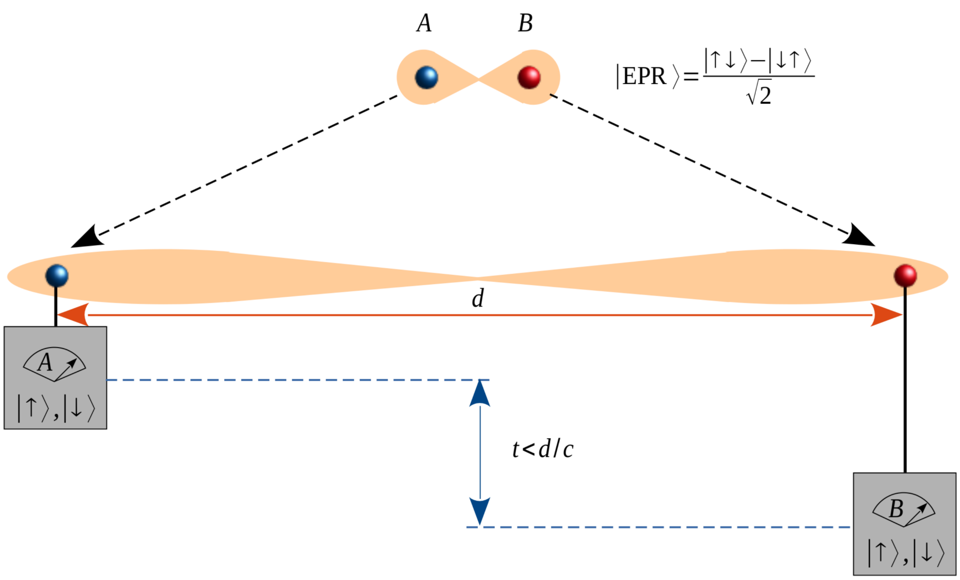


Figure 2: *Sketch of the Einstein-Podolsky-Rosen gedankenexperiment: An EPR pair state is prepared locally. The two particles $A$ and $B$ are then spatially separated by a distance $d$ before being measured independently at their localization. The time delay $t$ between the measurements on $A$ and $B$ is smaller than the time needed by any physical signal to travel from $A$ to $B$, $d/c$.*

does not play with dice"). This does not mean at all that he did not trust the predictions of quantum physics, but in his view, there must exist a more fundamental theory that quantum physics is a limit case of, which should predict deterministically the outcome of a quantum experiment. The apparent randomness of quantum physics would then be of the same nature than in statistical mechanics: a consequence of the lack of exact knowledge of the system. Einstein tried to find a situation that indeed showed that the formulation of quantum physics could not be whole story. In 1935, in collaboration with Podolsky and Rosen, he presented a situation where they argued that, in their opinion, quantum mechanics is incomplete [1]. Their argument relied on pairs of particles whose position and momentum states are measured. Here we discuss the technically simpler version proposed by Bohm in 1952, involving entangled spin states (see Fig. 2). Assume you create an EPR pair:

$$|\mathrm{EPR}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B + |\uparrow\rangle_B |\downarrow\rangle_A\right) , \qquad (22)$$

and separate the two particles $A$ and $B$ by a distance $d$ using a process that does not affect the spins. As we know, measuring the spin of $A$, you get $|\uparrow\rangle_A$ with probability $1/2$ and $|\downarrow\rangle_A$ with probability $1/2$: the outcome is totally random. However, measuring $|\uparrow\rangle_A$ on $A$, the spin of $B$ is instantaneously projected onto $|\downarrow\rangle_B$ and, if you get $|\downarrow\rangle_A$, the spin of $B$ is instantaneously projected onto $|\uparrow\rangle_B$. Thus, measuring the state of $B$ completely determine the result of the measurement performed on $A$. According to

Einstein, Podolsky and Rosen there are only two possibilities for this to occur: (i) either some information about the outcome of the measurement on $A$ is transmitted to $B$ in a time smaller than the time interval between the measurements performed on $A$ and $B$, or (ii) the result of the measurements ($A, B$ in $|\uparrow\rangle$ or $|\downarrow\rangle$) has been fixed at the creation of the pair by some kind of hidden process. The first explanation (i) can not be true as it would violate special relativity: the information cannot travel faster than the speed of light $c$ and, as one can separate the measurements on $A$ and $B$ by a time $t$ such that $t < d/c$, no physical signal can be transmitted from $A$ to $B$ in between the two measurements. Hence, the only logical conclusion is that there must exist some *local hiden variable* that determines the results of the measurement before $A$ and $B$ are separated.

Let us rephrase this argument using classical images to see that the conclusion of EPR about the existence of local hidden variables governing the outcome of a measurement is in fact quite natural. Consider two coins with two sides, head ($H$) and tail ($T$). Assume you want to understand the correlations in a state $|HH\rangle + |TT\rangle$. It would mean that you don't know whether you will obtain $H$ or $T$ following the tossing of the coin, but that you will aways obtain the following random outcomes in a series of toss: $HH, HH, TT, HH, TT, TT....$ This is a hint that the coins are not independent, otherwise you would obtain events like $HT$ or $TH$. But this is easy to imagine a mechanism that connects a $H$ with a $H$ or a $T$ to a $T$, like a small magnet inside the coins or some dips in the faces that pairs $H$ with $H$. This is the mechanism that EPR had in mind: a common variable that governs the correlations during the preparation of the state.

When the EPR article came out, Bohr understood that this was a very serious attack of the quantum theory. A debate then started between the proponents of hidden variable theories and those of the orthodox interpretation of quantum mechanics, known as the *Copenhagen interpretation* [2, 3, 4, 5]. Once again, Einstein, Podolsky, and Rosen did not question quantum mechanics itself nor its predictive power. They only argue that it must be complemented by the introduction of yet unknown (hidden) variables to understand the origin of the correlations in an entangled state. Contrarily, Bohr thought that there was no need for this: the strong correlations pre-exist in the state and we cannot assign any well-defined state to any subpart *before* we perform the measurement. The debate was thus rather about the interpretation of quantum mechanics and has remained philosophical for a long time.

**Bell's inequalities.** In 1964, John S. Bell proposed a situation, testable in real experiments, able to distinguish between local hidden variable (LHV) theory and the quantum theory [6]. He showed that some correlations have a bound lower in LHV theories than in quantum theory. This yields inequalities, known as the *Bell's inequalities*, that must be satisfied by LHV theories but not by quantum mechanics.

In the following, we do not use the original formulation by Bell based on the configuration of Einstein, Podolski, and Rosen, but instead use spin states to derive the Clauser, Horne, Shimony, Holt inequalities [7]. Assume that some hidden variable $\lambda$ determines the values of the spins $A$ and $B$. We make no hypothesis on $\lambda$: It can be a scalar or a vector, discrete or continuous, ... We do not know its value nor how it is

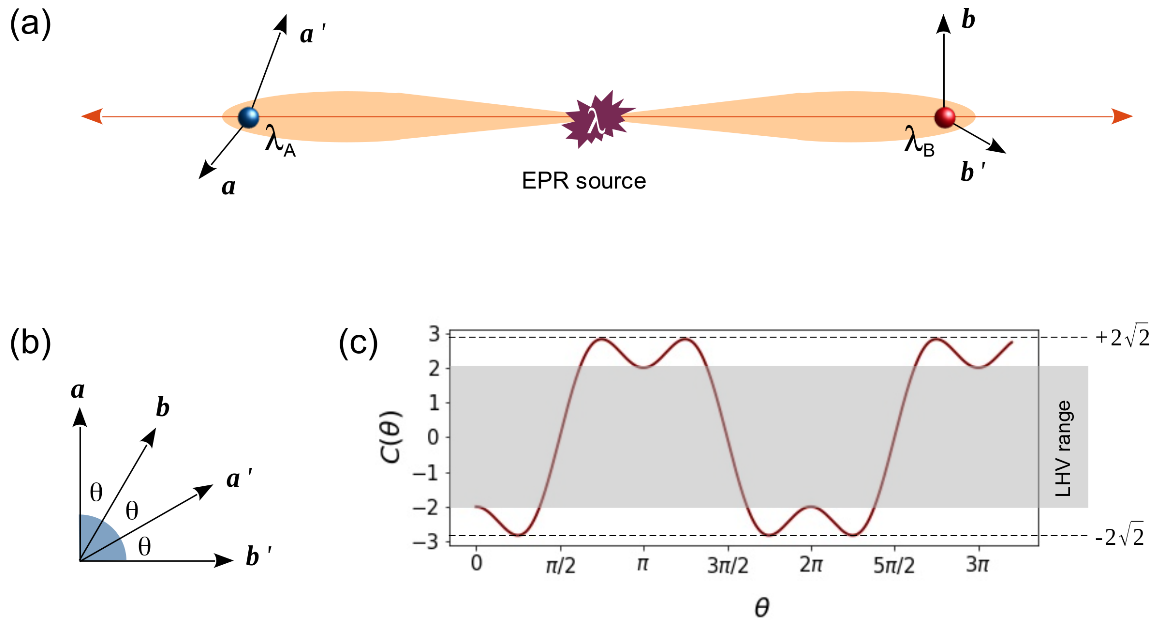


Figure 3: *Bell inequalities. (a) General scheme of the Bell experiment. An EPR pair of entangled 1/2-spin particles is emitted by a source, one of the particles going to the left (A) and the other to the right (B). The spin of particle A is then measured in one of the directions* $\boldsymbol{a}$ *or* $\boldsymbol{a}'$*, and the spin of particle B in one of the directions* $\boldsymbol{b}$ *or* $\boldsymbol{b}'$*. In the framework of a LHV theory, it is assumed that a hidden variable* $\lambda$ *has been created at the time of emission, each of the particles carrying a copy of this variable,* $\lambda_A$ *or* $\lambda_B$*. (b) Orientation of the directions* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}'$*, and* $\boldsymbol{b}'$ *of the spin measurements of the two particles used to establish the Bell's inequality (26). (c) Bell's correlation function as a function of the relative angle of the measurement directions in the framework of the quantum theory (solid red line) and range allowed in the framework of an LHV theory (shaded area).*

generated by Nature, but we assume it is decided at the creation of the pair, so that $A$ and $B$ can each carry a copy. The spin of each particle is a function of the *local* measurement direction $u$ and of the LHV $\lambda$, which has to vary randomly from shot-to-shot to explain the apparent randomness of the outcome of the measurement. It takes only two possible values:

$$\sigma_A(\boldsymbol{u}, \lambda) = \pm 1 \qquad \text{and} \qquad \sigma_B(\boldsymbol{u}, \lambda) = \pm 1 \ , \tag{23}$$

where the value of $\lambda$ decides whether it is $+1$ or $-1$ for each particle and each measurement direction. Now, choose two measurement directions for each particle, $\boldsymbol{a}$ and $\boldsymbol{a}'$ for $A$, and $\boldsymbol{b}$, and $\boldsymbol{b}'$ for $B$, (see Fig. 3a), and introduce the expectation value

$$E(\boldsymbol{u}, \boldsymbol{v}) = \sigma_A(\boldsymbol{u}, \lambda) \cdot \sigma_B(\boldsymbol{v}, \lambda) \ . \tag{24}$$

The key insight consists in introducing the correlation function:

$$\begin{aligned} C(\lambda) &= E(\boldsymbol{a},\boldsymbol{b}) - E(\boldsymbol{a},\boldsymbol{b}') + E(\boldsymbol{a}',\boldsymbol{b}) + E(\boldsymbol{a}',\boldsymbol{b}') \\ &= \sigma_A\big(\sigma_B - \sigma_B'\big) + \sigma_A'\big(\sigma_B + \sigma_B'\big) \ . \end{aligned} \tag{25}$$

As $\sigma_B - \sigma_B'$ and $\sigma_B + \sigma_B'$ take the two correlated values 0 or 2, this correlation function can only take the values $C(\lambda) = \pm 2$, irrespective of the LHV and the measurement directions. Hence, whatever the distribution $P(\lambda)$ of $\lambda$ from shot-to-shot, the average correlation function is bounded:

$$-2 \leq \langle C \rangle_{\text{LHV}} = \int d\lambda\, P(\lambda)\, C(\lambda) \leq 2 \ . \tag{26}$$

Strikingly, we never had to specify the details of the LHV theory, simply its existence! The Bell's inequalities are therefore very general.

What is the prediction of quantum physics? The quantum counterpart of the Bell's correlation function is

$$\hat{C} = \hat{E}(\boldsymbol{a},\boldsymbol{b}) - \hat{E}(\boldsymbol{a},\boldsymbol{b}') + \hat{E}(\boldsymbol{a}',\boldsymbol{b}) + \hat{E}(\boldsymbol{a}',\boldsymbol{b}') \ , \tag{27}$$

with $\hat{E}(\boldsymbol{u},\boldsymbol{v}) = \big(\hat{\boldsymbol{\sigma}}_A\cdot\boldsymbol{u}\big)\otimes\big(\hat{\boldsymbol{\sigma}}_B\cdot\boldsymbol{v}\big)$ and $\hat{\boldsymbol{\sigma}}_j$ the vector spin operator for particle $j \in \{A, B\}$. Assume now that the two spins are prepared in the EPR state, and the measurement directions $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{b}'$, and $\boldsymbol{b}'$ are chosen so that they make successive angles of $\theta$, see Fig. 3(b). The quantum average value of the Bell's correlator is then (see exercise 2):

$$-2\sqrt{2} \leq \langle C \rangle_{\text{Q}} = \cos(3\theta) - 3\cos(\theta) \leq 2\sqrt{2} \ . \tag{28}$$

Remarkably, the quantum correlations can reach values as large as $2\sqrt{2}$, thus violating the Bell's inequality (26) for some values of the angle $\theta$. This offers an experimentally-measurable quantity to distinguish LHV theories from quantum theory.

**Experimental tests.** The first conclusive experimental violations of Bell's inequalities were carried out in the United States in 1972-1976, in particular by John Clauser. They used polarizations of photons rather than spins. The typical scheme of a Bell's experiment is shown in Fig. 4(a), where the choice of the measurement directions is made by the polarizer $A$ and $B$, and the correlation functions are analyzed using a coincidence detector [8]. The first challenge was the production of entangled photon pairs. In Clauser's experiment, they were produced with a two-photon radiative cascade, similar to the one discussed in Fig. 1(b), that produces an EPR-like pair of two photons at slightly different frequencies, see Fig. 4(b). The Bell's inequality used reads

$$\delta = \left| \frac{R(22.5°)}{R_0} - \frac{R(67.5°)}{R_0} \right| - \frac{1}{4} \ , \tag{29}$$

where $R(\phi)$ is some coincidence function, which depends on the angle $\phi$ between the detectors[1]. The main result is a measure of $R(\phi)$, shown on Fig. 4(c) versus $\phi$. The

[1]There is a factor of two between the relative angle $\phi$ of the polarizers and the Bloch sphere angle $\theta$. For instance, orthogonal polarizations ($\phi = \pi/2$) correspond to antiparallel Bloch vectors ($\theta = \pi$). Hence the maximum violation of the Bell's inequalities is expected at $\phi = \pi/8 = 22.5°$.

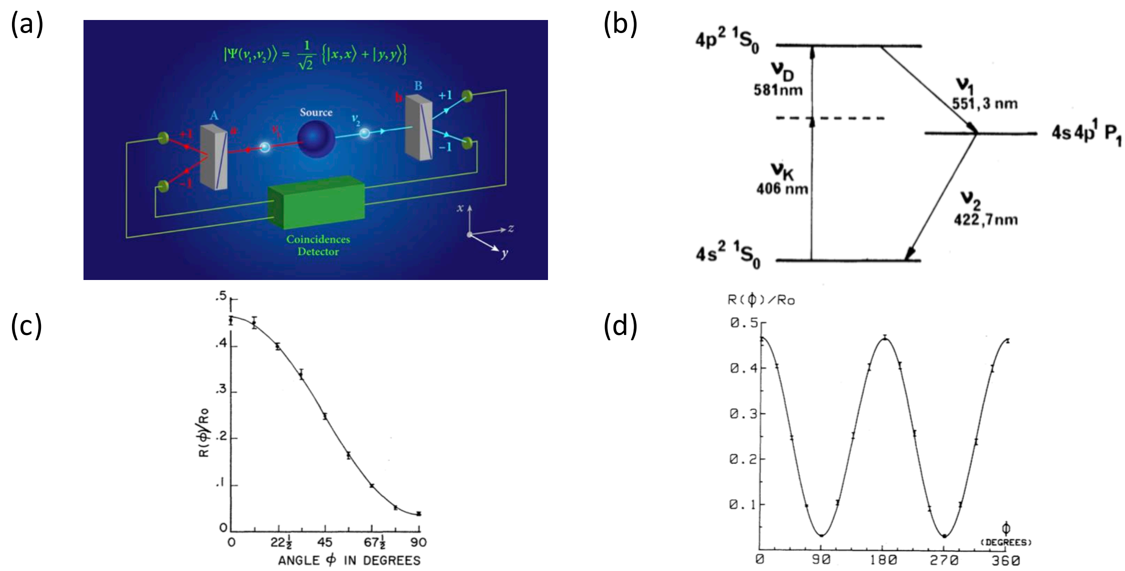


Figure 4: *Experimental tests of Bell's inequality. (a) Typical experimental scheme using photons: A source (central blue sphere) produces a pair of entangled photons at frequencies $\nu_1$ (red) and $\nu_2$ (light blue). Each passes through a polarizer (grey boxes A and B), which deflects it by a polarization-dependent angle. The output photons in either polarization state are detected by single-photon detectors (green dots), and analyzed by a coincidence detector (green box). From Ref. [4]. (b) Two-photon cascade in Ca atoms, producing the entangled photon pair. From Ref. [8]. (c) and (d) Experimental measurements of the $R(\phi)$ function defined in Eq. (29), from Refs. [8] and [9], respectively.*

experimental measurements (dots) are in excellent agreement with the prediction of quantum theory (solid line). In particular, they yield

$$\delta \simeq 0.05 \pm 0.008\ , \tag{30}$$

showing a violation of the Bell's inequality (29) by about 6 standard deviations. A similar experiment was performed by Fry and Thomson, leading to a similar conclusion with an even better resolution [10].

The Freedman and Clauser experiment is clearly in favor of quantum physics. It is, however, not free of experimental imperfections, which leaves *loopholes* to LHV theories. They are three of them:

(i) Polarizers loophole : Use of imperfect polarizers (most pairs were lost);

(ii) Detection loophole : Use of imperfect detectors (many pairs were undetected);

(iii) Locality loophole : Measurement axes chosen before the emission of the pairs.

The loopholes (i) and (ii) refer to possible experimental biases. Since most of the data is lost, the statistics may be biased, hence erroneously invalidating the Bell's inequalities.

In contrast, the loophole (iii) raises a fundamental question: Since the choice of the detection axes is made well before the photon pairs are emitted, it cannot be excluded that the emission process is affected by the detectors. It may then produce non-entangled pairs with a statistics depending on the detection axes such that the correlation function agrees with the quantum theory, while the communication channel is still subluminal.

The loopholes (i) and (iii) were lifted in a series of three experiments by Alain Aspect in the early 1980's. The first experiment was very close to Clauser's, but it made use of a source with a higher pair emission rate [11]. This led to a better measurement (Fig. 4d): $\delta \simeq 0.0572 \pm 0.00043$, a violation by more than 10 standard deviations. However, the experiment used single-channel polarizers, which allowed only one polarization to pass at a time. To overcome this issue, the second experiment was performed with double-channel polarizers, thus avoiding such a loss of data [9]. The Bell's correlator used in this experiment is that of Eq. (25). The experiment yields

$$C(22.5°) \simeq 2.697 \pm 0.015\ . \tag{31}$$

The quantum theory predicts for the specific experimental arrangement $C(22.5°) \simeq 2.70 \pm 0.05$, while the Bell's inequality stipulates $-2 \leq C(\theta) \leq 2$. The prediction differs from the theoretical value $2\sqrt{2} \simeq 2.83$ due to two effects: (i) the imperfections of the polarizers, and (ii) the finite solid angles of the detectors, which include contributions of photons in non exactly opposite directions. Taking into accounts these effects, the experiment thus violates Bell's inequality and confirms the quantum theory. Finally, the third experiment aimed at lifting loophole (iii). To do this, Aspect used a quasiperiodic sequence to determine the orientation of the polarizers [12]. The interest of this experiment lies in the fact that the choice of polarization is made quasi-randomly and faster than the time of flight of the photons from the source to the polarizers. Hence, the choice of the measurement axes is posterior to the emission of the photon pair so that the emission process cannot be affected by the prior choice of the measurements made. Here again, the experiment shows a clear violation of the Bell's inequalities, with about 5 standard deviations. All together, these experiments provide convincing evidence of the violation of the Bell's inequalities and the nonlocal character of quantum physics.

For purists, some loopholes remained: the detection loophole, as well as the fact that the quasiperiodic sequence of choices of the measurement directions was not strictly random. The sequence was indeed quasi-deterministic and chosen before the emission. The experimental choice could thus – in principle – influence the emission of the photon pairs. and lead to an artificial violation of the Bell's inequalities. The experiments were thus refined until the middle of the 2010's to remove any doubt. For instance, the group of Anton Zeillinger used a truly random number generator, based itself on a quantum measurement [13]. This lifted the criticism of the quasiperiodic sequence previously used. Only in the early 2010's did the detectors become efficient enough to detect up to 90% of the pairs produced. This allowed to lift the detection loophole [14, 15]. Finally, the last criticism was that the different loopholes had been lifted in different experiments but never simultaneously. This was finally done in three different experiments performed in 2015 [16, 17, 18].

**Implications of the violation of the Bell's inequalities.** Their violation validates the existence of the non-local quantum correlations. It leads to the abandon of local

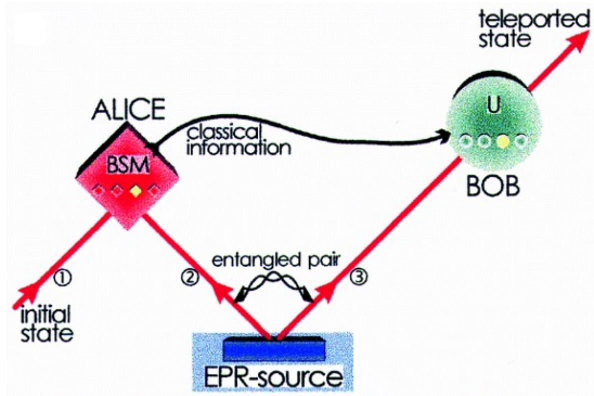


Figure 5: *Principle of the teleportation experiment. Figure from [19].*

realism dear to Einstein: it is not possible to assign a well defined state to a part of a system independently of the other parts. It is quite counter-intuitive, but so it is... It shows once more that we find obvious about the world is not necessarily correct...

Another important consequence is that a measurement on a part $A$ of a system may affect another part $B$. Indeed, the measurement on $A$ projects, not only the state of $A$, but also the full state of the bipartite system onto the eigenspace corresponding to the result of the measurement. This phenomenon is exploited in several applications such as *quantum teleportation* (see Sec. 5) and *quantum nondemolition measurements* (see lecture 5).

# 5 Quantum teleportation

We finally discuss an application where entanglement is used as a resource for quantum communications. The teleportation protocol allows one to transmit a quantum state between two distant systems. The principle of teleportation in science fiction consists in transferring an object from a point $A$ to a distant point $B$. In doing so, the object disappears from point $A$ and reappears at point $B$. This is exactly what quantum teleportation does, except that it is not an object that is teleported, but a state.

**Quantum teleportation protocol.** The teleportation protocol, shown in Fig. 5, was proposed by Bennett [20]. Alice has a qubit $A$ in an arbitrary state $|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$, which she may or may not know, and that she wishes to transmit to Bob. Bob has two qubits, $B$ (the receiver) and $M$ (the mediator). It is not necessary that the three qubits are of the same physical nature. For instance, one can be a photon and the other two ions, or any other combination.

To initiate the teleportation protocol, Bob prepares the qubit pair $MB$ in a maximally entangled state, for example: $|\phi_-\rangle_{MB} = (|01\rangle_{MB} - |10\rangle_{MB})/\sqrt{2}$. Bob keeps the qubit $B$ and sends the qubit $M$ to Alice. The three-qubit state is:

$$|\Psi\rangle_{AMB} = \big(\alpha|0\rangle + \beta|1\rangle\big)_A \otimes \frac{1}{\sqrt{2}}(|01\rangle_{MB} - |10\rangle_{MB}) \, . \tag{32}$$

Alice then performs a Bell's measurement on the qubit pair $AM$, *i.e.* a measurement able to distinguish between the four Bell states (see exercise A.3). In order to find the possible measurement results, we rewrite the three-qubit state using the Bell's basis for the two-qubit system $AM$ and the computational basis for the qubit $B$ (check it!):

$$
\begin{aligned}
|\Psi\rangle_{AMB} \; = \; & \frac{1}{2}\Big[ |\psi_+\rangle_{AM} \otimes \big(\alpha\,|1\rangle - \beta\,|0\rangle\big)_B + |\phi_+\rangle_{AM} \otimes \big(-\alpha\,|0\rangle + \beta\,|1\rangle\big)_B \\
& + |\psi_-\rangle_{AM} \otimes \big(\alpha\,|1\rangle + \beta\,|0\rangle\big)_B + |\phi_-\rangle_{AM} \otimes \big(-\alpha\,|0\rangle - \beta\,|1\rangle\big)_B \Big] \, .
\end{aligned} \tag{33}
$$

After the measurement, Alice obtains one of the following results: $m = (00)$, $m = (10)$, $m = (01)$ or $m = (11)$ corresponding to $|\psi_+\rangle$, $|\psi_-\rangle$, $|\phi_+\rangle$ or $|\phi_-\rangle$, each with a probability 1/4. The state of the qubit $B$ is then projected onto one of the states $|\psi'_m\rangle_B$ with

$$
\begin{aligned}
|\psi'_{00}\rangle = \alpha\,|1\rangle - \beta\,|0\rangle \;\; , \;\; |\psi'_{01}\rangle = -\alpha\,|0\rangle + \beta\,|1\rangle \;\; , \\
|\psi'_{10}\rangle = \alpha\,|1\rangle + \beta\,|0\rangle \;\; , \;\; |\psi'_{11}\rangle = -\alpha\,|0\rangle - \beta\,|1\rangle \;\; .
\end{aligned} \tag{34}
$$

Having obtained one of the Bell states after the measurement, Alice loses all information of her initial state, contained in the coefficients $\alpha$ and $\beta$. In turn, the full quantum information of the initial state is found in the state of Bob's qubit, who obtains a state that depends on these same coefficients. This information is, however, still hidden, as the state obtained on the qubit $B$ depends on the result of the measurement performed by Alice, which is random and which Bob ignores. Moreover, the state Bob obtains is not always exactly the initial one of Alice. To complete the teleportation, Bob applies a unitary operation $\hat{U}_m$ to his qubit, in order to restore the initial state of Alice's qubit on his own qubit.

$$
\begin{aligned}
\hat{U}_{00} = \hat{Y} \;\; &: \;\; \alpha\,|1\rangle - \beta\,|0\rangle \;\to\; -i\big(\alpha\,|0\rangle + \beta\,|1\rangle\big) \\
\hat{U}_{01} = \hat{Z} \;\; &: \;\; -\alpha\,|0\rangle + \beta\,|1\rangle \;\to\; -\big(\alpha\,|0\rangle + \beta\,|1\rangle\big) \\
\hat{U}_{10} = \hat{X} \;\; &: \;\; \alpha\,|1\rangle + \beta\,|0\rangle \;\to\; \alpha\,|0\rangle + \beta\,|1\rangle \\
\hat{U}_{11} = \hat{\mathbb{1}} \;\; &: \;\; \alpha\,|0\rangle + \beta\,|1\rangle \;\; .
\end{aligned}
$$

In all cases, the state obtained after the application of this unitary operation on $B$ is indeed the initial qubit state of $A$, up to a phase. However, the operation $\hat{U}_m$ to be performed by Bob depends on the result $m$ of Alice's measurement. Hence, Alice must transfer her result to Bob: This information is transmitted by a classical channel. Knowing the result of the measurement performed by Alice, Bob can then apply the appropriate operation. This concludes the teleportation process.

A few remarks are in order. First quantum teleportation is not incompatible with the no-cloning theorem. Indeed, $B$ has recovered the complete information initially contained in $A$ without anyone – neither Alice or Bob – having any knowledge of this information, *i.e.* of the initial state of $A$. More importantly, Alice loses all information about the initial state of her qubit: She finds one of the 4 Bell states, each with the same probability (1/4), completely independently of the coefficients $\alpha$ and $\beta$. Alice can therefore no longer make any measurements that would inform her about their initial values, even partially. Second, the transfer of quantum information requires both a quantum channel and a classical channel. In the case considered here, the latter is the transmission by Alice of her measurement result to Bob. Hence, the transmission cannot be faster than light. Third, the use of maximally entangled states is necessary: not maximally entangled states alters the fidelity of the transmission [20].

Finally, quantum teleportation has applications in quantum computing. For instance, it allows to transfer – in principle perfectly – any quantum information contained in a processor to another processor or to a memory. The protocol does not require that the qubits $A$, $M$, and $B$ be of the same physical nature. This allows to transfer information between different platforms, hence creating so-called *interconnects*. This is useful to implement complex quantum protocols by distributing each task on the most appropriate architecture. For instance, trapped ions perform very well on gate fidelities but suffer from decoherence. Conversely, photons are not well adapted to the implementation of two-qubit gates but are easier to isolate from their environment, and therefore less prone to decoherence. Using teleportation, we may then perform calculations on ionic qubits and then store the information on photonic qubits. The information stored in the photonic qubits can then be transferred back to the ionic qubits at any time to perform further calculations.

**Experimental demonstrations.** The first experimental realization of quantum teleportation used photonic systems [19]. It was, however, incomplete because the reconstruction process could not be implemented. Complete quantum teleportation was first demonstrated in ion traps [21, 22].

# A Problem set for Lecture 3

## A.1 Bell's inequalities

1. Consider the EPR state

$$|\text{EPR}\rangle = \frac{|{\uparrow}\rangle_A |{\downarrow}\rangle_B - |{\downarrow}\rangle_A |{\uparrow}\rangle_B}{\sqrt{2}} \ . \tag{35}$$

   (a) Recall that the eigenstates of the spin operator along the axis oriented by the vector $\boldsymbol{u}$, characterized by the spherical angles $\theta$ and $\varphi$ are

$$|{\uparrow}\boldsymbol{u}\rangle = \cos(\theta/2)e^{-i\varphi/2}\,|{\uparrow}_z\rangle + \sin(\theta/2)e^{i\varphi/2}\,|{\downarrow}_z\rangle \tag{36}$$
$$|{\downarrow}\boldsymbol{u}\rangle = -\sin(\theta/2)e^{-i\varphi/2}\,|{\uparrow}_z\rangle + \cos(\theta/2)e^{i\varphi/2}\,|{\downarrow}_z\rangle \ . \tag{37}$$

   Write the state $|\text{EPR}\rangle$ in the basis $\{|\varepsilon_{\boldsymbol{a}}\rangle \otimes |\varepsilon_{\boldsymbol{b}}\rangle\}$ with $\varepsilon = \uparrow$ or $\downarrow$. For simplicity, take $\varphi_A = \varphi_B$.

   (b) Conclude that the EPR state is the same for any quantization axis, provided the same is used for both particles $A$ and $B$.

2. We aim at deriving the quantum average of the Bell's correlator, Eq. (28). Consider the EPR state (35) and assume simultaneous measurement of the spin components $\hat{\sigma}_{Aa} = \hat{\boldsymbol{\sigma}}_A \cdot \boldsymbol{a}$ and $\hat{\sigma}_{Bb} = \hat{\boldsymbol{\sigma}}_B \cdot \boldsymbol{b}$.

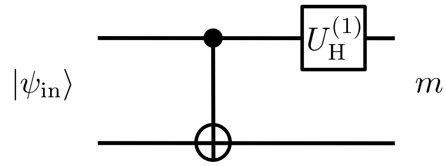


Figure 6: *Quantum circuit implementing a Bell measurement.*

(a) What are the possible outcomes and with what probabilities are they found? Use the result of 1(b).

(b) What are the possible results if we measure the spin of only one particle and with what probabilities are they found?

(c) What is the conditional probability that the measurement on particle $B$ gives the result $+$, assuming that the measurement on particle $A$ has given $-$?

(d) It is assumed (only here) that $\boldsymbol{a} = \boldsymbol{b}$. Show that the result of the measurement of one spin is perfectly determined by that of the other spin. What can be said about the correlation between the results of these measurements?

(e) Show that the quantity $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = \langle \hat{\sigma}_{1\boldsymbol{a}} \hat{\sigma}_{2\boldsymbol{b}} \rangle$ may be written as $E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b}$. Use the probabilities found above.

(f) The vector $\boldsymbol{a}$ being fixed, we consider the situation where the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}'$, and $\boldsymbol{b}'$ form the same angle $\theta$ with the previous vector, see Fig. 3(b). Find the expression of the Bell correlation function

$$\langle C \rangle_{\rm Q} = E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}) - E_{\rm Q}(\boldsymbol{a}, \boldsymbol{b}') + E_{\rm Q}(\boldsymbol{a}', \boldsymbol{b}) + E_{\rm Q}(\boldsymbol{a}', \boldsymbol{b}') \tag{38}$$

as a function of the angle $\theta$.

(g) Find the extrema of the correlation function $\langle C \rangle_{\rm Q}$ versus $\theta$ and show that there is a conflict between quantum theory and local hidden variable theories.

3. Show that for any product state of the spins $A$ and $B$, the average value of the quantum Bell correlator, Eq. (27), fulfills the classical Bell inequality, $|\langle C \rangle_{\rm Q}| \leq 2$. You may use $\langle \psi | \hat{\boldsymbol{\sigma}} | \psi \rangle = \boldsymbol{\psi}$, where $\hat{\boldsymbol{\sigma}}$ is the vector spin operator and $\boldsymbol{\psi}$ is the Bloch vector associated to the ket $|\psi\rangle$.

## A.2 Quantum teleportation

Starting from Eq. (32), prove Eq. (33).

## A.3 Bell measurement

Show that the quantum circuit represented in Fig. 6 performs a Bell measurement, *i.e.* maps the input states $|\psi_+\rangle$, $|\psi_-\rangle$, $|\phi_+\rangle$ or $|\phi_-\rangle$ onto the classical outputs $m = (00)$, $m = (10)$, $m = (01)$ or $m = (11)$.

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