Master QLMN Antoine Browaeys
Physics of Quantum Information antoine.browaeys@institutoptique.fr

Lecture 1: review of quantum physics

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Quantum Physics relies on a small number of principles and concepts. These have been developed following a pretty involved inductive process during the first half of the 20th century, triggered and fueled by experimental findings. Here, we will ignore this historical process and simply state its outcome in terms of six principles. We will illustrate them on the example of two-level systems, which will play a central role in this course as carriers of quantum information.

1 Superposition, states and Hilbert space

The observation of interferences with individual photons and matter-waves made, for example, of electrons, neutrons or molecules led to the idea that the theory has to fulfill the superposition principle: quantum systems can be prepared in a superposition of several states or follow several paths. The mathematical framework naturally including linear superpositions is linear algebra. Besides, the existence of steady states, i.e. states that evolve in time but without change in their probability, requires the use of complex amplitudes in the superposition: Hilbert space, vector space based on $\mathbb{C}$, is then the appropriate mathematical frame to work with.

Principle 1 (states): the state of an *isolated* quantum system is *fully* characterized by a vector $|\psi\rangle$ of an Hilbert space $\mathcal{E}_{\rm H}$. This means that the state vector contains *all* the information available on the system.

Any Hilbert space, either of finite or infinite dimension, features countable orthonormal basis $\{|\phi_n\rangle\}$, i.e. basis whose elements are labelled by an integer number n which runs up to the dimension of the space (it can be infinite). An orthonormal basis fulfills $\langle\phi_n|\,\phi_m\rangle = \delta_{nm}$. These are called Hilbert basis. Any vector has a decomposition on such basis:

$$|\psi\rangle = \sum_{n=1}^{\dim\mathcal{E}_{\rm H}} a_n\,|\phi_n\rangle\,, a_n \in \mathbb{C}\ . \tag{1}$$

Besides, a ket $|\psi\rangle$ has an associated bra $\langle\psi|$. The hermitian product between two vectors $|\psi\rangle = \sum_n a_n\,|\phi_n\rangle$ and $|\chi\rangle = \sum_n b_n\,|\phi_n\rangle$ is the complex number defined as

$$\langle\psi|\,\chi\rangle = \sum_n a_n^* b_n\ . \tag{2}$$

Two-level systems and qubits. In this course, two kinds of quantum objects will play a central role: two-level systems and harmonic oscillators. As far as two-level systems are concerned, examples are many and will be discussed in details in the next lectures: two states of a spin 1/2, two internal states of an atom or a molecule selected by tuning

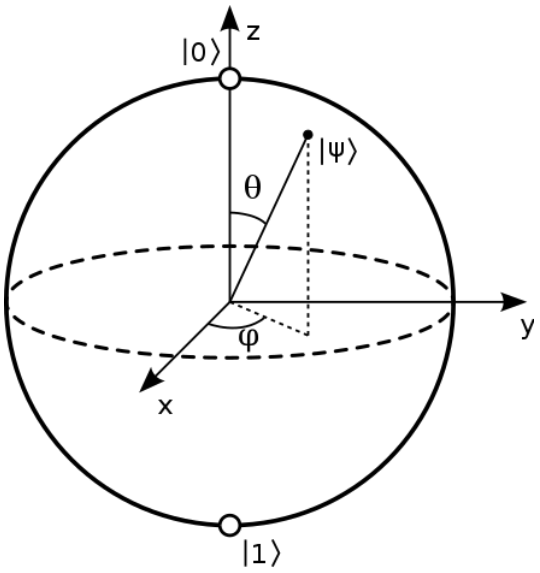

Figure 1: *Bloch vector pointing on the Bloch sphere (from Wikipedia).*

the frequency of a laser or microwave, quantum circuit with a non-linearity, double-well potential, polarization of a photon,... In the context of quantum information, the two-states are labeled $|0\rangle$ and $|1\rangle$ and they form a basis of a Hilbert space of dimension 2. The superposition of the two is called a quantum bit or "qubit" and its general expression for any normalized state vector is

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,e^{i\varphi}\,|1\rangle \ . \tag{3}$$

Hence, the state is fully determined by two angles θ and φ, in the same way any unit vector $\mathbf{u}$ in $\mathbb{R}^3$ is defined by its spherical coordinates. This mapping between the qubit state and a vector on a unit sphere allows one to represent a superposition state, as shown in Fig. 1: in this context, the vector $\mathbf{u}(\theta,\varphi)$ is called the "Bloch vector" and the sphere is named the "Bloch sphere". For example, the states $|0\rangle$ and $|1\rangle$ are represented by the two poles, while the state $(|0\rangle+|1\rangle)/\sqrt{2}$ points along x and $(|0\rangle+i\,|1\rangle)/\sqrt{2}$ along y. Keep in mind that even if we use here the vocabulary of quantum information, all the properties of qubits are generic of *any* two-level systems.

The link between the hermitian product of two states $|\psi\rangle$ and $|\psi'\rangle$ and the scalar product of the two corresponding Bloch vectors $\mathbf{u}$ and $\mathbf{u}'$ deserves some caution: taking $\varphi=\varphi'=0$, one finds $\langle\psi|\psi'\rangle = \cos[(\theta-\theta')/2]$ but $\mathbf{u}\cdot\mathbf{u}' = \cos(\theta-\theta')$. Hence, two orthogonal states have opposite (i.e. non orthogonal...) Bloch vectors.

2 Observables

Classically, any physical system is described by variables that one can measure, called observables. Examples are position, momentum, angular momentum, energy, dipoles, electric field... The corresponding quantum description is given by the second principle:

Principle 2 (observables): any observable A of a quantum system is described by an hermitian operator $\hat{A}$ acting in the Hilbert space describing the system.

Recall that hermitian implies the following relation between the matrix elements: $\langle\varphi_n|\,\hat{A}\,|\varphi_m\rangle = A_{nm} = A^*_{mn}$, i.e. the adjoint operator $\hat{A}^\dagger = \hat{A}$. Mathematically, there is a difference between auto-adjoint and hermitian for infinite dimension. As it has very little practical consequence, we will ignore this.

For any hermitian operator $\hat{A}$ there exists an orthonormal basis $\{|\phi_n\rangle\}$ where the operator is diagonal, the associated eigenvalues a_n being real. This important property is the reason for imposing that observables are hermitian operators: as we will see in the next section, any measurement on an observable yields one of its eigenvalues, which must be real for the theory to be meaningful. Imposing hermiticity for the operator automatically fulfills this. The *spectral decomposition* of the operator in terms of projectors follows:

$$\hat{A} = \sum_n a_n \hat{P}_n, \quad \text{with} \quad \hat{P}_n = |\phi_n\rangle\langle\phi_n| \ , \tag{4}$$

when the eigenvalues are not-degenerate. For degenerate eigenvalues (degeneracy g_n):

$$\hat{P}_n = \sum_{\alpha=1}^{g_n} |\phi_n^\alpha\rangle\langle\phi_n^\alpha| \ . \tag{5}$$

In both cases, $\sum_n \hat{P}_n = \hat{\mathrm{Id}}$. Finally, the average value of an observable for a quantum system in a state $|\psi\rangle$ is $\langle\hat{A}\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle$, and its variance $(\Delta A)^2 = \langle\psi|\,(\hat{A} - \langle A\rangle)^2\,|\psi\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 = ||(\hat{A} - \langle A\rangle)\,|\psi\rangle\,||^2$.

Case of a two-level system. The most general matrix expression of an observable acting in a Hilbert space of dimension 2 (basis $\{|0\rangle, |1\rangle\}$) is:

$$\hat{A} = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix} = \frac{a+d}{2}\hat{\mathrm{Id}} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

with a, b, c, d *real* numbers. Any observable can thus be decomposed on the identity operator and the three Pauli matrices $\hat{\sigma}_i$'s defined by:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{7}$$

Be aware that the Pauli operators have different notations in the litterature: $\sigma_{x,y,z} = \sigma_{1,2,3} = X, Y, Z$.

Properties of Pauli operators. Show that:

1. $\sigma_i^2 = \mathrm{Id}$,
2. $\mathrm{Tr}(\sigma_i) = 0$,
3. $[\sigma_x, \sigma_y] = 2i\sigma_z$ and all the circular permutations.
4. The eigenvalues of the σ_i's are ± 1

5. The eigenstates of the σ_i's are respectively $(|0\rangle \pm |1\rangle)/\sqrt{2}$, $(|0\rangle \pm i\,|1\rangle)/\sqrt{2}$, $(|0\rangle\,, |1\rangle)$ (represent them on the Bloch sphere).

Spin 1/2 and Bloch vector. Consider a spin-1/2 particle such as the electron. The spin operator is related to the Pauli matrices by $\hat{\mathbf{S}} = (\hbar/2)\boldsymbol{\sigma}$. Prepare the spin state of the electron in $|\psi\rangle = \cos\frac{\theta}{2}\,|\downarrow\rangle + \sin\frac{\theta}{2}\,e^{i\varphi}\,|\uparrow\rangle$, associated to the Bloch vector $\mathbf{u}(\theta,\varphi)$. Show that $\langle\psi|\hat{\mathbf{S}}|\psi\rangle = (\hbar/2)\mathbf{u}$. Hence, one way to visualize the Bloch vector of a qubit is to associate the qubit to a spin 1/2 and think about the direction of the average spin $\langle\hat{\mathbf{S}}\rangle$, or equivalently of its magnetic moment $\boldsymbol{\mu} \propto \langle\hat{\mathbf{S}}\rangle$.

3 Measurements

Any physical theory has to define rules on how to compare its predictions to the outcome of an experiment. Quantum physics has it special that the outcome of any experiment is probabilistic, and as far as we know today this is fundamental: contrarily to statistical physics where probabilities are used to hide our lack of knowledge about the details of the system, the result of a measurement in quantum physics is intrinsically random and we do not believe that we will ever find a more profound deterministic theory from which the quantum probabilities will emerge.

Generally speaking, we expect from a measurement of an observable $\hat{A}$ (classical or quantum) that if we repeat it a second time *immediately after* the first one we find the exact same result. Otherwise the measurement process is not meaningful. This means that if the state of the system after the first measurement is $|\psi_a\rangle$, the variance on the results for $\hat{A}$ following a second measurement performed right after is $(\Delta A)^2 = 0$, with an average value $\langle\hat{A}\rangle$. Therefore:

$$(\Delta A)^2 = ||(\hat{A} - \langle\hat{A}\rangle)\,|\psi_a\rangle\,||^2 = 0 \Rightarrow \hat{A}\,|\psi_a\rangle = \langle\hat{A}\rangle\,|\psi_a\rangle \ , \tag{8}$$

and the state $|\psi_a\rangle$ is an eigenstate of $\hat{A}$, the average value being an eigenvalue of $\hat{A}$. These considerations lead to the measurement principle.

Principle 3 (projective measurements): the result of a measurement of the observable $\hat{A}$ on a quantum system in a state $|\psi\rangle$ can only be one of the eigenvalues a_n of $\hat{A}$. The state of the system *immediately* after the measurement is the projection of $|\psi\rangle$ on the subspace $\mathcal{E}_n$ associated to the eigenvalue a_n:

$$|\psi\rangle' = \frac{\hat{P}_n\,|\psi\rangle}{||\hat{P}_n\,|\psi\rangle\,||} \ . \tag{9}$$

The probability to get the result a_n is $p_n = \langle\psi|\,\hat{P}_n\,|\psi\rangle$ (Born's rule). In the simplest case of a non-degenerate eigenvalue $p_n = |\,\langle\psi_n|\,\psi\rangle|^2$.

The average value of an observable $\hat{A}$ is then $\langle\hat{A}\rangle = \sum_n a_n p_n$, and follows also directly from the spectral decomposition (4).

Let us again emphasize how weird a quantum measurement process is. One would expect that by measuring a quantum system, one would get its state *before* the measurement. However this is not the case: the only thing we know with certainty is the

state *after* the measurement. This fact is the basis of the preparation of a quantum system in a given state by performing a projective measurement (initialization by measurement). The only information that we obtain about the state of the system *before* the measurement is the following: if, by measuring an observable $\hat{A}$, we obtain the result a_n associated to the eigenstate $|\phi_n\rangle$, this tells us that the state before the measurement is not orthogonal to $|\phi_n\rangle$. This does not look like a lot of information and yet, as we will see in the next lectures, this has fundamental consequences.

Quantum projection noise. To obtain experimentally the probabilities p_n, one repeats N times the same experiment and counts the number of times N_n one gets a_n. Then $N_n/N \to p_n$ for large N. However, an intrinsic error noise is associated to a measurement. Consider for example the qubit state $|\psi\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$. Each time one repeats the measurement on a system prepared in $|\psi\rangle$ one obtains randomly $|0\rangle$ (probability p) or $|1\rangle$ (probability $1-p$), and each outcome is independent from the previous one. This is exactly like flipping a coin or undergoing a random walk in one dimension. The distribution of results is given by the binomial distribution: after N repetitions of the measurement we obtain the result $|0\rangle$ N_0 times, with a probability $P_0 = \binom{N}{N_0} p^{N_0}(1-p)^{N-N_0}$. There, the probability p is estimated by $p \approx N_0/N$. The average value of N_0 is $\langle N_0\rangle \approx Np$ and the associated standard deviation is $\Delta N_0 = \sqrt{Np(1-p)}$. This deviation is called the quantum projection noise and is intrinsic to the measurement process.

Measurement on a qubit. Take a qubit defined by a Bloch vector $\mathbf{u}(\theta,\varphi)$, and measure its state along the axis $\mathbf{m}$. Show that $p_{0,1} = (1 \pm \mathbf{m}\cdot\mathbf{u})/2$.

Despite the apparent "simplicity" of the projective measurement postulate, we can already guess that it has to contain subtleties and that it can not be the full story. For example, when we say that the state of the system after the measurement is the eigenstate associated to the eigenvalue of a given observable, we implicitly assume that the system still exists after the measurement. But this is often not the case ! Consider for example the measurement of the position of an atom by ionization with a laser. After the measurement, the atom does not exist any longer, as it is now an ion, and the electron emitted is what we actually detect by electronic amplification: the state of the atom after ionization is then meaningless. This example shows that one needs to consider the role of the measuring device in a measurement process. We will come back to all these in the next lectures.

4 Evolution of quantum systems

In classical physics, the evolution of a system is described by "equations of motion" such as Newton's laws in mechanics or Maxwell's equations in electrodynamics. Usually, they can be derived from the Lagrangian or Hamiltonian of the system (which are actually inferred from the equations of motion...). This is also the case in quantum physics, where the knowledge of the Hamiltonian , i.e. the operator associated to the energy, is enough to calculate the evolution of the system. This is true even when including special relativity (Dirac equation). Of course, the Hamiltonian of a system may not be

easy to write, but it does exists as it is the operator associated to its energy...

Principle 4 (evolution): The evolution of the state of a *isolated* system is governed by the Schrödinger equation involving the Hamiltonian operator $\hat{H}(t)$:

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle \ . \tag{10}$$

A direct consequence of this equation and of the hermiticity of $\hat{H}(t)$ is the conservation of the norm of any state vector under a temporal evolution: $\langle\psi(t)|\,\psi(t)\rangle = \langle\psi(0)|\,\psi(0)\rangle$. This fact implies the existence of a unitary operator $\hat{U}(t)$, i.e. such that $\hat{U}^\dagger = U^{-1}$, transforming $|\psi(0)\rangle$ into $|\psi(t)\rangle$: $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$. The operator $\hat{U}(t)$ is called the evolution operator. When the Hamiltonian is time-independent, the evolution operator has a simple expression: $\hat{U}(t) = \exp[-i(\hat{H}/\hbar)t]$. As a consequence, the evolution of a state $|\psi(0)\rangle = \sum_n c_n\,|\phi_n\rangle$, with $H\,|\phi_n\rangle = E_n\,|\phi_n\rangle$, is

$$|\psi(t)\rangle = \sum_n c_n e^{-i\frac{E_n}{\hbar}t}\ |\phi_n\rangle \ . \tag{11}$$

In the general case of a time-dependent Hamiltonian, $\hat{U}(t)$ evolves according to:

$$i\hbar\frac{\partial\hat{U}(t)}{\partial t} = \hat{H}(t)\hat{U}(t) \ . \tag{12}$$

The temporal evolution of the average value of an operator $\hat{A}$, $\langle\hat{A}\rangle(t) = \langle\psi(t)|\hat{A}|\psi(t)\rangle$, is governed by Ehrenfest's theorem:

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \langle\frac{\partial\hat{A}}{\partial t}\rangle \tag{13}$$

Importantly, the temporal evolution of the state vector and of the average value of an observable is fully *deterministic*: the probabilistic nature of quantum physics mentioned above is only associated to the measurement process itself, not to the evolution before it. Besides, the measurement process described by the projective measurement postulate is generally speaking not reversible. This means that you can not construct a unitary operator $\hat{U}$ associated to a measurement. If you could, it would have to fulfill the following properties: for two arbitrary states $|\phi_{1,2}\rangle$, $\hat{U}\,|\phi_{1,2}\rangle = |\phi_{1,2}\rangle$ *and* $\hat{U}(|\phi_1\rangle+|\phi_2\rangle) = |\phi_1\rangle$ or $|\phi_2\rangle$. But this is not possible for a linear operator... The question of how to reconcile the unitary evolution of a superposition state with the existence of one (and only one...) particular outcome in an experiment is called the "measurement problem". It is still the subject of passionate debates, even if everyone agrees with the mathematical rules associated with the non-unitarity of a projective measurement.

Evolution of a two-level system: Rabi oscillations. Consider first the case of two states $|0\rangle$ and $|1\rangle$ separated by an energy $\hbar|\Delta|$, governed by an Hamiltonian

$$H = \frac{\hbar}{2}\begin{pmatrix}\Delta & \Omega\,e^{-i\varphi}\\ \Omega\,e^{i\varphi} & -\Delta\end{pmatrix} \ . \tag{14}$$

The quantity Ω is called the Rabi frequency. This form is actually general and applies to many situations such as a spin 1/2 placed in a magnetic field with a longitudinal and transverse component or the driving of a two-level atom by an electromagnetic field in the rotating-wave approximation (see Exercise A.2 and Lecture 2). If the system is initially prepared in $|\psi(0)\rangle = |0\rangle$, the probability p_1 to measure it in $|1\rangle$ after an evolution time t is given by the Rabi formula:

$$p_1(t) = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2\left(\sqrt{\Omega^2 + \Delta^2}\frac{t}{2}\right) \tag{15}$$

To demonstrate this expression, decompose $|\psi(0)\rangle$ onto the two eigenstates $|\pm\rangle$ of H with energies $\pm\hbar\sqrt{\Omega^2 + \Delta^2}/2$ and evolve the state according to Eq. (11).

To interpret this evolution in terms of a motion on the Bloch sphere, let us recall that the Bloch vector is $\mathbf{u} = \langle\boldsymbol{\sigma}\rangle$. Applying Ehrenfest's theorem, one gets (for $\varphi = 0$)

$$\frac{d\langle\boldsymbol{\sigma}\rangle}{dt} = \boldsymbol{\Omega} \times \langle\boldsymbol{\sigma}\rangle \quad \text{with} \quad \boldsymbol{\Omega} = \begin{pmatrix} \Omega \\ 0 \\ \Delta \end{pmatrix} \ . \tag{16}$$

Hence, the Bloch vector precesses around the vector $\boldsymbol{\Omega}$, at a rate $|\boldsymbol{\Omega}| = \Omega_{\rm R} = \sqrt{\Omega^2 + \Delta^2}$.

Let us consider now the important case where $\Delta = 0$, which gives

$$H = \frac{\hbar\Omega}{2}\left(\cos\varphi\,\hat{\sigma}_x + \sin\varphi\,\hat{\sigma}_y\right) \Rightarrow \hat{U}(t) = \cos\frac{\Omega t}{2}\,\hat{\rm Id} - i\sin\frac{\Omega t}{2}\left(\cos\varphi\,\hat{\sigma}_x + \sin\varphi\,\hat{\sigma}_y\right) \tag{17}$$

Therefore, starting from the state $|0\rangle$, we obtain

$$|\psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|0\rangle - i\sin\left(\frac{\Omega t}{2}\right) e^{i\varphi}\,|1\rangle \ . \tag{18}$$

More generally, for $H = \frac{\hbar\Omega_{\rm R}}{2}\,\boldsymbol{\sigma}\cdot\mathbf{n}$, one gets:

$$\hat{U}(t) = \cos\left(\frac{\Omega_{\rm R} t}{2}\right)\hat{\rm Id} - i\sin\left(\frac{\Omega_{\rm R} t}{2}\right)\boldsymbol{\sigma}\cdot\mathbf{n} = \mathcal{R}_{\mathbf{n}}(\alpha) \ . \tag{19}$$

with $\mathbf{n} = \boldsymbol{\Omega}/\Omega_{\rm R}$ a unit a vector. This unitary operator is a rotation around the axis $\mathbf{n}$, with an angle $\alpha = \Omega t/2$.

Application to single qubit gates. The expression (19) allows one to calculate single-qubit gates commonly used when coherently manipulating qubits. These are (i) the Z-rotation: $\mathbf{n} = \mathbf{z}, \Omega t = \pi, \Delta = 0$ leading to $\hat{U} = -i\hat{Z}$; (ii) the X-rotation: $\mathbf{n} = \mathbf{x}, \Omega t = \pi, \Delta = 0$ leading to $\hat{U} = -i\hat{X}$; (iii) the Hadamard gate $\mathbf{n} = (\mathbf{x}+\mathbf{z})/\sqrt{2}, \Omega = \Delta, \Omega_{\rm R} t = \pi$ leading to:

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ . \tag{20}$$

5 Tensor product states and entanglement

All quantum systems are in fact composed of several subparts that interact with each other: simply think of an atom consisting of electrons, protons and neutrons. More generally, an isolated system is always an idealization of a real-life situation where the quantum system is coupled (even if it is very weakly) to an environment, which we have to describe to understand the behavior of the system. Also, in describing a measurement process, one has to consider the system coupled to the measuring device (we will come back at length to these questions later in the course). A single system can also have several degrees of freedom: position and spin for atomic particles, photons with a propagation mode and a polarization mode, photon with two propagation modes... Each of theses modes, degrees of freedom or subparts are described by an Hilbert space, and the question is how to combine them to describe the whole system.

Principle 5 (tensor states): The Hilbert space describing a composite system $A+B$ is the tensor product of the Hilbert spaces describing each: $\mathcal{E}_{\rm H} = \mathcal{E}_A \otimes \mathcal{E}_B$.

The basis of $\mathcal{E}_H$ is the tensor product of the basis of $\mathcal{E}_A$ ($\{|a\rangle\}$) and $\mathcal{E}_B$ ($\{|b\rangle\}$): $\{|a\rangle \otimes |b\rangle\}$. As a consequence, $\dim\mathcal{E}_H = \dim\mathcal{E}_A \times \dim\mathcal{E}_B$. Any vector of $\mathcal{E}_H$ has the form:

$$|\psi\rangle = \sum_{a=1}^{\dim\mathcal{E}_A} \sum_{b=1}^{\dim\mathcal{E}_B} c_{ab} \, |a\rangle \otimes |b\rangle \ . \tag{21}$$

Importantly most of them are *not* of the form $|\phi_A\rangle \otimes |\chi_B\rangle$. Tensor states are often written in a simplified way $|\phi_A\rangle \otimes |\chi_B\rangle = |\phi_A, \chi_B\rangle$ (be careful with the order!).

Case of two-level systems. Take two qubits of the form $|\phi_A\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and $|\chi_B\rangle = \gamma\,|0\rangle + \delta\,|1\rangle$. Then:

$$|\phi_A\rangle \otimes |\chi_B\rangle = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \ . \tag{22}$$

The dimension of the Hilbert space is 4. Had we considered N qubits, the tensor Hilbert space would have had a dimension 2^N, i.e. an exponential scaling with the number of constituents.

Operators in tensor spaces. By definition:

$$[\hat{A} \otimes \hat{B}]\,|\phi_A\rangle \otimes |\chi_B\rangle = [\hat{A}\,|\phi_A\rangle] \otimes [\hat{B}\,|\phi_B\rangle] \tag{23}$$

As an example, the matrix of the tensor operator is thus:

$$\text{for } \hat{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ then } \hat{A} \otimes \hat{B} = \begin{pmatrix} \alpha\hat{B} & \beta\hat{B} \\ \gamma\hat{B} & \delta\hat{B} \end{pmatrix} \ . \tag{24}$$

Measurement with tensor states. The projective measurement principle 3 applies here. A subtlety arises when one is only interested in a measurement on a subpart of

the system. Assume you measure an observable $\hat{A}$ of a composite system $A+B$. The probability to obtain the result a (one of the eigenstates of $\hat{A}$) is obtained by applying the principle 3 using the projector: $\hat{P}_a = |a\rangle\langle a| \otimes \hat{\mathrm{Id}}_B$, indicating that one measures on A, leaving B untouched.

Entanglement. As we noticed above, most of the states of a composite system $A+B$ are not factorizable in the form $|\phi_A\rangle \otimes |\chi_B\rangle$. Any state that can not be written in this form, i.e. non factorizable, is said to be entangled.

We will devote a lot of time in this course to entanglement, but let us first take a simple example for two qubits. Show first that the states $(|00\rangle \pm |11\rangle)\sqrt{2}$, and $(|01\rangle \pm |10\rangle)\sqrt{2}$ are entangled. These four states form a basis of the Hilbert space describing the two qubits. They are call Bell's states.

The main property of entangled states is the existence of strong non-local correlations between their constituents. Take again the Bell state $(|00\rangle + |11\rangle)\sqrt{2}$ and show that: $p(A:0) = 1/2$, $p(A:0, B:0) = 1/2$, but that the conditional probability $p(B:0|A:0) = 1$. Hence, when prepared in this Bell state, the outcome of measurements on A and B are perfectly correlated. Although already weird, this alone is not unknown to the classical world: two classical objects possessing a given property can be correlated. Take for example two balls, one red the other one blue. If I hand you the red one, you know for sure that I have the blue. However, in quantum physics, the violation of Bell's inequalities (see Lecture 3) tells us that the quantum correlations are stronger than the classical ones.

The experimental preparation of high quality entangled states is very challenging and is one of the main endeavor for experimentalists working on quantum technologies. Besides, the characterization of entanglement is even more challenging. We will discuss this at several occasions during the course. As for the generation of entangled states, it can be obtained by combining the single qubit gates that we described in Sec. 4 and so-called two-qubit gates. These gates feature two inputs and two outputs. One input is called the target and the other one the control. Assume for example that you know how to realize a CNOT (controlled NOT) gate whose matrix is given by:

$$U^{(2)}_{\mathrm{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{25}$$

Combine the Hadamard gate and the CNOT gate to generate the Bell's states. We will see in Lecture 2 how to realize a CNOT gate in practice.

No-cloning theorem. As an application of tensor states, let us demonstrate a theorem with implications for quantum communication. It was introduced in the early 80's by Wootters and Zurek and states that one can not copy an unknown quantum state $|\psi\rangle$ without destroying it. Assume you could. This would mean that if you have two qubits, one in state $|\psi\rangle$, the other in state $|0\rangle$, you could construct a unitary operator $\hat{U}$ such that $\hat{U}(|\psi\rangle\,|0\rangle) = |\psi\rangle\,|\psi\rangle$. By considering two non-orthogonal states $|\psi\rangle$ and $|\phi\rangle$ and the fact that $\hat{U}$ conserves the hermitian product, show that this is not possible

Identical particles: fermions and bosons. The last principle of quantum physics

deals with the fact that all particles that we know of fall in two categories, called fermions and bosons. Their defining property is that their state behaves differently in a permutation of two of them when they are identical, i.e. their quantum numbers are all the same (spin, position, momentum, charge, mass, angular momentum...).

Principle 6 (symmetrization): The quantum state $|\psi(1,2)\rangle$ of two *identical* bosons is symmetric under the permutation of the two, i.e. $|\psi(1,2)\rangle = |\psi(2,1)\rangle$. The quantum state $|\psi(1,2)\rangle$ of two *identical* fermions is anti-symmetric under the exchange of the two, i.e. $|\psi(1,2)\rangle = -|\psi(2,1)\rangle$.

This distinction between fermions and bosons is only relevant when you consider at least two of them, as you need at least two particles to apply a permutation. This is why we discuss it in the section dealing with composite systems and tensor product. As an example, take two orthogonal states $|a\rangle$ and $|b\rangle$ and two particles 1 and 2. A possible two-atom state is $|1:a\rangle \otimes |2:b\rangle$. This is a perfectly suitable state if the two particles are *not* identical. However this is not a physically acceptable state to describe two *identical* particles: the symmetrization principle states that of all the possible tensor states you can form, only a very limited subset is possible, namely the ones symmetric and anti-symmetric under a permutation. Hence, here, $|\psi_{\mathrm{B}}(1,2)\rangle = (|1:a;2:b\rangle + |1:a;2:b\rangle)/\sqrt{2}$ for bosons and $|\psi_{\mathrm{F}}(1,2)\rangle = (|1:a;2:b\rangle - |1:a;2:b\rangle)/\sqrt{2}$ for fermions.

It turns out also that fermions carry semi-integer spin $(1/2, 3/2, 5/2...)$, while bosons have integer spins $(0, 1, 2, 3...)$. This is a consequence of the spin-statistics theorem, but even more, this is an experimental fact... Fundamental constituents of matter (electrons, protons, neutrons) are fermions with spin 1/2, while particles mediating fundamental interactions (such as photons, gluons, $\mathrm{W}^{\pm}$, Z^0) are bosons. Assembling an odd number of fermions makes a fermionic particle, while assembling an even number makes a bosonic one. This is why atoms can be either bosons or fermions depending on the isotope.

References: J.L. Basdevant, J. Dalibard, "Mécanique quantique", Ellipse

A Problem set for Lecture 1

A.1 Some derivations...

Properties of Pauli operators. Show that:

1. $\sigma_i^2 = \mathrm{Id}$,
2. $\mathrm{Tr}(\sigma_i) = 0$,
3. $[\sigma_x, \sigma_y] = 2i\sigma_z$ and all the circular permutations.
4. The eigenvalues of the σ_i's are ± 1
5. Find the corresponding eigenstates of $\hat{\sigma}_i$'s. Represent them on the Bloch sphere.

Spin 1/2 and Bloch vector. Consider a spin-1/2 particle such as the electron. The spin operator is related to the Pauli matrices by $\hat{\mathbf{S}} = (\hbar/2)\boldsymbol{\sigma}$. Take a unit vector $\mathbf{u}$ with spherical coordinates (θ, φ) and construct the operator $\hat{S}_{\mathbf{u}} = \hat{\mathbf{S}} \cdot \mathbf{u}$.

1. Calculate the eigenvalues and eigenvectors of $\hat{S}_{\mathbf{u}}$.
2. Show that ${}_{\mathbf{u}}\langle +|\hat{\mathbf{S}}|+\rangle_{\mathbf{u}} = (\hbar/2)\mathbf{u}$.

Measurement on a qubit. Take a qubit defined by a Bloch vector $\mathbf{u}(\theta, \varphi)$, and measure its state along the axis $\mathbf{m}$. Show that $p_{0,1} = (1 \pm \mathbf{m} \cdot \mathbf{u})/2$.

Ehrenfest's theorem. Demonstrate the theorem.

Evolution operator. Take $H = (\hbar\Omega/2)\hat{\sigma}_x$ and demonstrate that the corresponding evolution operator is:

$$\hat{U}(t) = \cos\left(\frac{\Omega t}{2}\right) \hat{\mathbf{Id}} - i \sin\left(\frac{\Omega t}{2}\right) \hat{\sigma}_x \ . \tag{26}$$

Rotation operator. Take $H = (\hbar\Omega/2)\hat{\sigma}_z$ and assume the initial state of a qubit to be $|\psi(0)\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle$. Calculate the state $|\psi(t)\rangle$ and show that the evolution corresponds to a rotation of the Bloch vector around the $\hat{z}$ axis with angle Ωt.

A.2 Rabi oscillations (very important: part of the lecture!)

Consider two states $|0\rangle$ and $|1\rangle$ separated by an energy $\hbar\omega_0$. The qubit is driven by a coupling oscillating in time with the frequency ω: think of a two-level atom driven my a laser or a spin 1/2 placed in a rotating magnetic field. The Hamiltonian is:

$$H = -\frac{\hbar\omega_0}{2}\,\hat{\sigma}_z + \hbar\Omega\,\cos\omega t\,\hat{\sigma}_x = \hbar \begin{pmatrix} -\omega_0/2 & \Omega\,\cos\omega t \\ \Omega\,\cos\omega t & \omega_0/2 \end{pmatrix}_{|0\rangle,|1\rangle} . \tag{27}$$

We want to calculate the state $|\psi(t)\rangle$ of the qubit after an evolution time t. As the Hamiltonian is time-dependent, the usual method leading to Eq. (11) does not work. However, we will see that using a unitary transform, equivalent to working in the frame rotating at the frequency ω, and a near-resonant approximation (called the *rotating wave approximation*), the Hamiltonian can be made time-independent.

1. Introduce the state $|\tilde{\psi}(t)\rangle = \hat{\mathcal{R}}(t)\,|\psi(t)\rangle$ with

$$\mathcal{R}(t) = \exp[-i\frac{\omega t}{2}\hat{\sigma}_z] = \begin{pmatrix} e^{-i\frac{\omega t}{2}} & 0 \\ 0 & e^{i\frac{\omega t}{2}} \end{pmatrix}_{|0\rangle,|1\rangle} . \tag{28}$$

Using the general expression of a qubit (Eq. 3) to explain why this transformation amounts to applying a rotation around the Oz axis of angle $-\omega t$, and "to move to the frame rotating at frequency ω".

2. Show that the new states $|\tilde{\psi}(t)\rangle$ is solution of the new Schrödinger equation

$$i\hbar\frac{d}{dt}\,|\tilde{\psi}(t)\rangle = \tilde{H}(t)\,|\tilde{\psi}(t)\rangle \quad \text{with} \quad \tilde{H} = \mathcal{R}H\mathcal{R}^{-1} + i\hbar\frac{d\mathcal{R}}{dt}\mathcal{R}^{-1} . \tag{29}$$

3. Show that:

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega\,(1+e^{-2i\omega t}) \\ \Omega\,(1+e^{2i\omega t}) & -\Delta \end{pmatrix}_{|0\rangle,|1\rangle} , \tag{30}$$

with $\Delta = \omega - \omega_0$ the detuning.

4. This new hamiltonian is still time-dependent. However, the terms $e^{\pm 2i\omega t}$ oscillate rapidly with respect to all the other frequency scales Δ or Ω in the problem, and can therefore be neglected. Apart from a numerical simulation to check this, it is not so easy to justify precisely this fact. However, the following hand-wavy argument helps. Take $|\tilde{\psi}\rangle = c_0(t)\,|0\rangle + c_1(t)\,|1\rangle$. The Schrödinger equation (29) reads

$$\dot{c}_0 = -i\frac{\Delta}{2}c_0 - i\frac{\Omega}{2}(1+e^{-2i\omega t})c_1 \tag{31}$$

$$\dot{c}_1 = i\frac{\Delta}{2}c_1 - i\frac{\Omega}{2}(1+e^{2i\omega t})c_0 . \tag{32}$$

If $\Omega \ll |\Delta|$, the second equation gives $c_1(t) = \alpha e^{i\Delta t/2}$ (α is a constant) and the first equation becomes

$$\dot{c}_0 \sim -i\frac{\Delta}{2}c_0 - i\frac{\Omega}{2}(1+e^{-2i\omega t})e^{i\frac{\Delta t}{2}}\alpha , \tag{33}$$

which can be solved by using the variation of constant method. Write $c_0(t) = A(t)\,e^{-i\Delta t/2}$. Then $\dot{A} = -i\frac{\Omega}{2}(1+e^{-2i\omega t})e^{i\Delta t}\alpha$, and

$$A(t) \sim -i\frac{\Omega}{2}\left[\frac{e^{i\Delta t}-1}{i(\omega-\omega_0)} - \frac{e^{-i(\omega+\omega_0)t}-1}{i(\omega+\omega_0)}\right] . \tag{34}$$

Explain then why, in the near resonant case $|\omega - \omega_0| \ll \omega + \omega_0$, one can neglect the rapidly oscillating factor $e^{\pm 2i\omega t}$ in $\tilde{H}$.

5. We are now back to a time-independent Hamiltonian

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix} . \tag{35}$$

Calculate the eigenenergies of $\tilde{H}$, and the eigenstates as a function of θ, where we have used: $\cos\theta = \Delta/\sqrt{\Omega^2+\Delta^2}$ and $\sin\theta = \Omega/\sqrt{\Omega^2+\Delta^2}$.

6. At $t = 0$ the qubit is in state $|0\rangle$. Calculate $|\tilde{\psi}(t)\rangle$.

7. Calculate the probability $p_1(t)$ to find the qubit in state $|1\rangle$ after a time t, and recover the Rabi formula

8. Plot $p_1(t)$ for $\Delta = 0$ and $\Delta = 3\Omega$.

9. Plot the envelop $p_1^{\max}$ of $p_1(t)$ as a function of Δ. Comment.

A.3 Inhibiting the evolution by measuring: the quantum Zeno effect

Consider the evolution of a qubit initially in $|0\rangle$, described by the Rabi formula with $\Delta = 0$.

1. What is the probability to find the qubit in $|0\rangle$ after a time $\delta t = T/N$ for $N \gg 1$?

2. What is the probability to still find the system in $|0\rangle$ after N consecutive measurements performed every δt ?

3. What is the probability to find the qubit in $|0\rangle$ after a time T when $N \to \infty$?

4. Compare to the case where you let the system evolve without measuring between $t = 0$ and $t = T$. Explain the effect in terms of the quantum measurement process.

A.4 Entangling gates

Assume you know how to realize the two-qubit π-phase gate represented by the matrix:

$$U_\pi^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{36}$$

1. Prepare the target and control qubits in $(|0\rangle + |1\rangle)\sqrt{2}$. Calculate the two-qubit state at the output.

2. Is it an entangled state and why ?

3. Consider the elementary quantum circuit shown in Fig. 2. Show that it is equivalent to a CNOT gate.

4. Take N target qubits and one control qubit. Draw a circuit involving CNOT gates only allowing you to prepare a $N + 1$ Greenberger-Horne-Zeilinger state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}} \left(|0, 0, 0...\rangle + |1, 1, 1...\rangle \right) . \tag{37}$$

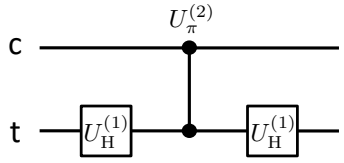

Figure 2: *Quantum circuit to produce a CNOT gate from a π-Phase gate. Here $U_{\mathrm{H}}^{(1)}$ is the Hadamard gate; c, t are the control and target qubits.*