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Quantum gates

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In the perspective of building a quantum computer, physicists have developed techniques to manipulate the state of quantum bits. These are called quantum gates and a quantum computer will be a machine running sequences of such gates. We study here a few examples of important gates that we will also use to illustrate some concepts in the coming lectures.

1 Single-qubit gates

These operations act on a single qubit. They rely on the driving of the qubit (transition frequency ω_0) by a classical field at a frequency ω.

1. An X-gate consists of a $\pi/2$-rotation of the qubit state around the Ox axis of the Bloch sphere. Write the matrix $\hat{U}_X$ acting on the qubit state.
2. Take one example of physical implementation discussed in the lecture and explain (in less than 5 lines) how such a gate can be realized.
3. If the Rabi frequency corresponding to the coupling between the qubit and field is $\Omega/(2\pi) = 1$ MHz, what is the duration of the gate ?
4. The Hadamard gate is defined by the matrix :

$$\hat{U}_{\mathrm{H}}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ . \tag{1}$$

   Show that it corresponds to a rotation of the Bloch vector around $\mathbf{n} = (\mathbf{x}+\mathbf{z})/\sqrt{2}$, How do you have to choose Ω with respect to $\Delta = \omega - \omega_0$. What is the duration of the gate ?
5. Estimate the laser intensity necessary to drive an optical dipole transition with a Rabi frequency of $\Omega/(2\pi) = 10$ MHz.
6. Estimate the magnetic field of a microwave necessary to drive the hyperfine transition of Rb atom at 6.8 GHz with a Rabi frequency $\Omega/(2\pi) = 1$ kHz.

2 Entangling gates

These gates operate on two qubits, the first being called target, the second control. Assume you know how to realize the two-qubit π-phase gate represented by the matrix in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis :

$$U_{\pi}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{2}$$

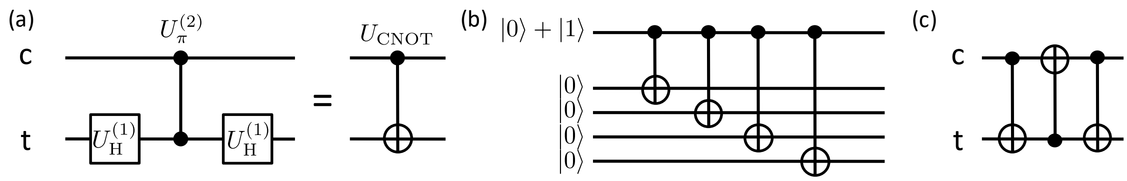

FIGURE 1 – *(a) Quantum circuit to produce a CNOT gate from a π-Phase gate. Here $U_{\rm H}^{(1)}$ is the Hadamard gate; c, t are the control and target qubits. (b) Quantum circuit to prepare a Greenberger-Horne-Zeilinger state. (c) Swap gate consisting of three CNOT gates.*

1. Prepare the target and control qubits in $(|0\rangle + |1\rangle)\sqrt{2}$. Calculate the two-qubit state at the output.
2. Is it an entangled state and why ?
3. Consider the elementary quantum circuit shown in Fig. 1(a). Show that it is equivalent to a CNOT gate, whose matrix is :

$$U_{\rm CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3}$$

4. Consider the circuit shown in Fig. 1(b). What is the state at the output ?
5. **Swap gate**. Consider finally the circuit represented in Fig. 1(c) and show that it swaps the states of the control and target qubit.

3 Examples of implementation of entangling gates

1. **Gate with neutral atoms.** Consider two atoms trapped in optical tweezers and distant by $R = 5\,\mu$m. When they are in their ground state $|g\rangle = |0\rangle$, they do not interact at such a distance. When excited to a state $|r\rangle = |1\rangle$ with very large principal quantum number n (Rydberg state), their interaction is considerably enhanced and is of the form $V = C_6/R^6$.
   (a) Explain why the corresponding Hamiltonian is $\hat{H} = V\hat{n}_1 \otimes \hat{n}_2$, with $\hat{n}_i = (1 + \sigma_i^z)/2$.
   (b) Show that this Hamiltonian can realize a two-qubit π-phase gate an give its duration T.
2. **Gate with superconducting circuits.** Consider two superconducting circuits as described in the lecture, each with states $|0\rangle$ and $|1\rangle$ (frequency ω_0), coupled to a microwave cavity with resonant frequency ω. We have seen in the Lecture on Approximation Methods (example H) that the two qubits exchanging virtually a photon via the cavity are described by the Hamiltonian :

$$\hat{H} = J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) \quad \text{with} \quad J = \frac{\hbar\Omega^2}{\Delta} . \tag{4}$$

   Write the matrix of the gate (called iSWAP) for $JT/\hbar = \pi$. Using single gates acting on the qubits, this iSWAP can generate a CNOT gate.