Homework 2 - Quantum gates Solution.

I Single-qbit gates

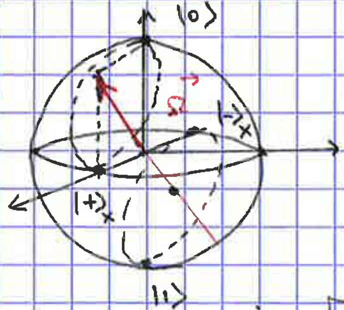
1. From equation 19 in lecture 1 with $\frac{\Omega t}{2} = \frac{\pi}{4}$ and $\varphi = 0$

$$U(t) = \frac{1}{\sqrt{2}} \mathbb{1} - i \frac{1}{\sqrt{2}} \sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

2. Two hyperfine states of alkali atom driven by a microwave, or by a Raman transition.

3. For $\Omega T = \pi/2 \Rightarrow T = \frac{\pi}{2} \frac{1}{2 \times 2\pi \times 10^6} = 250$ ns.

4. The gate transforms $|0\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle_x$, and $|1\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle_x$



In order to have $\vec{\Omega} = \begin{pmatrix} \Omega \\ 0 \\ \Delta \end{pmatrix}$ aligned along $\hat{x} + \hat{z}$ you need to choose $\Omega = \Delta$

Hence the total Rabi frequency is $\sqrt{\Omega^2 + \Delta^2} = \Omega_1$ and $T = \frac{\pi}{2\Omega_1}$

5. $\Omega = \frac{dE_L}{\hbar}$ where d is the amplitude of an optical dipole.

$d \sim e a_0 \sim 10^{-29}$ C.m and for a plane wave modelling a laser $I_L = \frac{1}{2} \epsilon_0 c E_L^2$

$$\Rightarrow I_L = \frac{1}{2} \epsilon_0 c \left(\frac{\hbar \Omega}{d}\right)^2 = \frac{1}{2} 10^{-11} \times 3 \times 10^8 \left(\frac{6 \times 10^{-34} \times 10^7}{10^{-29}}\right)^2 \simeq 500 \text{ W/cm}^2$$

Hence 0,5 mW focused on 1 mm^2.

6. $\frac{\mu_B B}{\hbar} = \Omega \Rightarrow B = \frac{\hbar \Omega}{\mu_B} = \frac{10^{-34} \times 6 \times 10^3}{10^{-23}} \simeq 6 \times 10^{-8}$ T

II

1. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \longrightarrow \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

2. not entangled: you can not write $|00\rangle + |01\rangle + |10\rangle - |11\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$

otherwise you would have: $\alpha\gamma = \alpha\delta = \beta\gamma = +1$ and $\beta\delta = -1$.

$\Rightarrow \alpha\beta\gamma\delta = +1$ and $\alpha\beta\gamma\delta = -1$ Absurde

3. $$U_{CNOT} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{U_H} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{U_H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4. $(|0\rangle + |1\rangle) \otimes |0\rangle |0\rangle |0\rangle |0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|00000\rangle + |11111\rangle)$

which is a 5 qbit GHZ state.

5- Look at the action of the circuit on the basis 00, 01, 10, 11

$$
\begin{array}{ccccccc}
00 & \rightarrow & 00 & \rightarrow & 00 & \rightarrow & 00 \\
01 & \rightarrow & 01 & \rightarrow & 11 & \rightarrow & 10 \\
10 & \rightarrow & 11 & \rightarrow & 01 & \rightarrow & 01 \\
11 & \rightarrow & 10 & \rightarrow & 10 & \rightarrow & 11 .
\end{array}
$$

Hence indeed swap the state of c and t.

III Implementation of a gate to entangle.

1. ⚠ In the text there was a mistake as the σ^z was defined differently from the lecture...

a) The atoms only interact when they are both in the state $|1\rangle$, and the interaction is C_6/R^6.

Defining $n_i = \frac{1-\sigma_i^z}{2} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]_{|0\rangle,|1\rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{|0\rangle,|1\rangle}$

Then $H = \frac{C_6}{R^6}\hat{n}_1\hat{n}_2$: only when $n_1 = n_2 = 1$ do the atoms interact, and the interaction energy is $C_6/R^6 = V$

b) The unitary operator acting on 2 qbits in a state $|\psi\rangle$ is

$U(t) = e^{-iHt/\hbar}$

In the basis $|00\rangle\,|01\rangle\,|10\rangle\,|11\rangle$: $U(t) = \begin{pmatrix} e^{-i0t/\hbar} & 0 & 0 & 0 \\ 0 & e^{-i0t/\hbar} & 0 & 0 \\ 0 & 0 & e^{-i0t/\hbar} & 0 \\ 0 & 0 & 0 & e^{-iVt/\hbar} \end{pmatrix}$

for $Vt/\hbar = \pi$ $U(t)$ realizes a CZ gate. $U_\pi^{(2)}$

2- $U(t) = e^{-iHt/\hbar}$. The Hamiltonian does not affect state $|00\rangle$ and $|11\rangle$

But exchanges the states $|01\rangle$ and $|10\rangle$, which are degenerate.

In this basis: $H = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}_{|01\rangle,\,|10\rangle}$, which is exactly the same as the one associated to a Rabi oscillation.

One can thus use Eq 19 of lecture 1 to calculate the unitary operator associated to the evolution: $U(t) = \cos\left[Jt/\hbar\right]\hat{\mathbb{1}} - i\sin\left[Jt/\hbar\right]\underset{\sim}{\hat{\sigma}_x}$

acting on $|\tilde{0}\rangle = |01\rangle$
$|\tilde{1}\rangle = |10\rangle$

The iSwap is obtained for $Jt/\hbar = \frac{\pi}{2}$ (again error in text)

$$\Rightarrow U_{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$