HW1 - Correction ; Playing with 2-level systems

A.1 Derivations

Pauli Matrices: take $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{|0\rangle, |1\rangle}$; $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{|0\rangle, |1\rangle}$

1. $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$

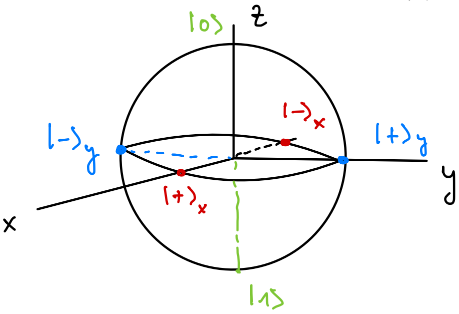
2. $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_x = 0$

3. $[\sigma_x, \sigma_y] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_z$

   Hence $[\sigma_x\, \sigma_y] = 2i\sigma_z$; $[\sigma_z, \sigma_x] = 2i\sigma_y$; $[\sigma_y, \sigma_z] = 2i\sigma_x$

4. As $\hat{\sigma}_i^2 = 1$, their eigenvalues are such that $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

5. $|\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$; $|\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i\,|1\rangle\right)$; $|\pm\rangle_z = |0\rangle ; |1\rangle$



Spin ½ and Bloch vector

Take $\vec{u} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \Rightarrow \vec{\hat{S}}\cdot\vec{u} = \hat{S}_u = S_x\hat{u}_x + S_y\hat{u}_y + S_z\hat{u}_z$

$$S_{\hat{u}} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix}$$

1. Calculate $(\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta \Rightarrow \lambda^2 = \cos^2\theta + \sin^2\theta = 1$

   $\lambda = 1 \Rightarrow$ Equation of subspace $(\cos\theta - 1)x + \sin\theta\, e^{-i\varphi}\, y = 0$

   eigenvector $\parallel \begin{pmatrix} -\sin\theta\, e^{-i\varphi} \\ \cos\theta - 1 \end{pmatrix} \parallel \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{pmatrix}$

   as $\sin\theta = 2\sin\theta/2\,\cos\theta/2$ and $1 - \cos\theta = 2\sin^2\theta/2$

   $|+\rangle_u = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,e^{i\varphi}\,|1\rangle$

And similarily: $|-\rangle_u = \sin\theta/_2 \, |0\rangle - \cos\theta/_2 \, e^{i\varphi} \, |1\rangle$

2.

$${}_u\langle + | \vec{S} | + \rangle_u = \frac{\hbar}{2} \begin{bmatrix} {}_u\langle + | \sigma_x | + \rangle_u \\ {}_u\langle + | \sigma_y | + \rangle_u \\ {}_u\langle + | \sigma_z | + \rangle_u \end{bmatrix}$$

And : $${}_u\langle + | \sigma_x | + \rangle_u = \left(\cos\theta/_2 \quad \sin\theta/_2 \, e^{-i\varphi} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta/_2 \\ \sin\theta/_2 \, e^{i\varphi} \end{pmatrix}$$

$$= 2\cos\frac{\theta}{2} \sin\theta/_2 \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2} \right) = \sin\theta \cos\varphi$$

Same calculations yield: $$\langle \vec{S} \rangle_u = \frac{\hbar}{2} \begin{bmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{bmatrix} = \frac{\hbar}{2} \vec{u}.$$

Measurement of a qbit :

take $\vec{m} = \vec{e}_z$

The state vector associated to $\vec{u} \, (\theta \, \varphi)$ is $|\psi\rangle = \cos\frac{\theta}{2} |0\rangle + \sin\theta/_2 \, e^{i\varphi} |1\rangle$

Hence $$p_0 = \cos^2\frac{\theta}{2} = \frac{1 + \cos\theta}{2} \quad \text{and} \quad p_1 = \sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}$$

As $\vec{m} \cdot \vec{u} = \cos\theta \Rightarrow p_{0,1} = \dfrac{1 \pm \vec{m} \cdot \vec{u}}{2}$

Ehrenfest's theorem :

$\langle A \rangle = \langle \psi(t) | A(t) | \psi(t) \rangle$

$$\frac{d}{dt}\langle A \rangle = \left[\frac{d}{dt} \langle \psi | \right] A |\psi\rangle + \langle \psi | \frac{\partial A}{\partial t} |\psi\rangle + \langle \psi | A \frac{d}{dt} |\psi\rangle$$

As $\dfrac{d}{dt}|\psi\rangle = \dfrac{1}{i\hbar} H |\psi\rangle$ and $\dfrac{d}{dt}\langle \psi | = -\dfrac{1}{i\hbar} H |\psi\rangle$

$$\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar} \langle \psi | AH - HA | \psi \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

$$= \frac{1}{i\hbar} \langle [A, H] \rangle + \left\langle \frac{\partial}{\partial t} A \right\rangle$$

Evolution operator

$$U(t) = \mathrm{Exp}\left[-i \, Ht/\hbar \right] = e^{-i \frac{\Omega t}{2} \sigma_x}$$

$$= \sum_{n=0}^{\infty} \frac{\left(-i \frac{\Omega t}{2}\right)^n}{n!} \sigma_x^n = \sum_{p=0}^{\infty} \left(-i \frac{\Omega t}{2}\right)^{2p} \sigma_x^{2p} \frac{1}{2p!} + \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \left(-i \frac{\Omega t}{2}\right)^{2p+1} \sigma_x^{2p+1}$$

Using $\sigma_x^{2p} = \hat{\mathbb{I}} \Rightarrow$ $$U(t) = \sum_{p=0}^{\infty} \left(\frac{\Omega t}{2}\right)^{2p} \frac{(-1)^p}{2p!} - i \sum_{p=0}^{\infty} \left(\frac{\Omega t}{2}\right)^{2p+1} \frac{(-1)^p}{(2p+1)!} \sigma_x$$

One recognizes the Taylor expansion of cos and sin.

hence $\hat{U}(t) = \cos\frac{\Omega t}{2}\,\hat{\mathbb{1}} - i\sin\frac{\Omega t}{2}\,\hat{\sigma}_x$

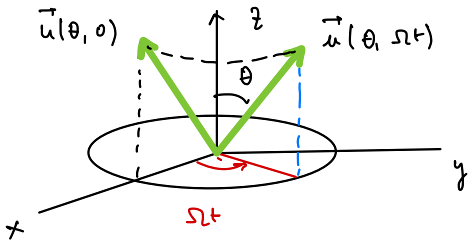
Rotation operator $\quad H = \frac{\hbar\Omega}{2}\sigma_z$ has two eigenvalues $\pm\frac{\hbar\Omega}{2}$

associated to the states $|0\rangle$ and $|1\rangle$

Thus $|\psi(0)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle \Rightarrow |\psi(t)\rangle = \cos\frac{\theta}{2}e^{-i\frac{\Omega t}{2}}|0\rangle + \sin\frac{\theta}{2}e^{i\frac{\Omega t}{2}}|1\rangle$

or, to an overall phase factor $|\psi(t)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\Omega t}|1\rangle$

One recognizes the general expression of a state associated to the Bloch vector $\vec{u}(\theta, \varphi = \Omega t)$, hence rotated by Ωt with respect to $|\psi(0)\rangle$



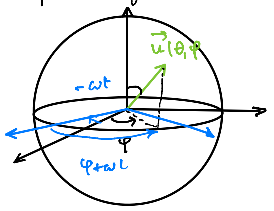
A.2 Rabi oscillations

1. If $|\psi(0)\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|1\rangle$

$\Rightarrow \quad |\psi(t)\rangle = e^{-i\,\omega t/2\,\hat{\sigma}_z}|\psi(0)\rangle = \cos\frac{\theta}{2}\,e^{-i\frac{\omega t}{2}}\,|0\rangle + \sin\frac{\theta}{2}\,e^{i\frac{\omega t}{2}}\,e^{i\varphi}|1\rangle$

To a phase factor: $|\psi(t)\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,e^{i(\varphi+\omega t)}\,|1\rangle$

Hence the new polar angle φ is $\varphi-(-\omega t)$: equivalent to a rotation of the frame by an angle $-\omega t$



2. $i\hbar\frac{d}{dt}R(t)\,|\psi(t)\rangle = i\hbar\,\frac{dR}{dt}\,R^{-1}\,|\tilde{\psi}\rangle + i\hbar\,R\,\frac{d}{dt}|\psi\rangle$

$= i\hbar\,\frac{dR}{dt}\,R^{-1}\,|\tilde{\psi}\rangle + R\,H\,R^{-1}\,|\tilde{\psi}\rangle$

hence: $\tilde{H} = RHR^{-1} + i\hbar\left[\frac{d}{dt}R\right]R^{-1}$

3. $i\hbar\left(\frac{d}{dt}R\right)R^{-1} = i\hbar\left(-i\frac{\omega t}{2}\,\sigma_z\right)\underbrace{RR^{-1}}_{=\mathbb{I}} = -\frac{\hbar\omega}{2}\,\sigma_z$

Thus $\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{pmatrix}\begin{pmatrix} -\omega_0 & \Omega\left(e^{i\omega t}+e^{-i\omega t}\right) \\ \Omega\left(e^{i\omega t}+e^{-i\omega t}\right) & \omega_0\end{pmatrix}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2}\end{pmatrix}$

$- \frac{\hbar}{2}\begin{pmatrix}\omega & 0 \\ 0 & -\omega\end{pmatrix}$

$= \frac{\hbar}{2}\begin{pmatrix}\omega-\omega_0 & \Omega\left(1+e^{-2i\omega t}\right) \\ \Omega\left(1+e^{2i\omega t}\right) & -\omega+\omega_0\end{pmatrix}$

4. Near-resonance: $|\omega-\omega_0| \ll \omega+\omega_0$, hence (*) $A(t) \sim -i\,\frac{\Omega}{2}\,\frac{e^{i\Delta t}-1}{i(\omega-\omega_0)} + o\left(\frac{\omega-\omega_0}{\omega+\omega_0}\right)$

Note: in practice $\omega-\omega_0 \sim$ few linewidths of an optical transition. (ex: in Rb @ 780 nm, $\Gamma \sim 2\pi\times 6\,\text{MHz}$). As $\omega_0 \sim 384$ THz, the approximation $\frac{\omega-\omega_0}{\omega+\omega_0} \sim \frac{\text{few }\Gamma}{2\omega_0} \ll 1$ is very well justified.

(*) is the result we would have obtained had we neglected $e^{\pm 2i\omega t}$ in $\tilde{H}$ from the start.

5. Writing $\tilde{H} = \frac{\hbar}{2}\begin{pmatrix}\Delta & \Omega \\ \Omega & -\Delta\end{pmatrix} = \frac{\hbar}{2}\sqrt{\Omega^2+\Delta^2}\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}$

we obtain $E_+ = +\frac{\hbar}{2}\sqrt{\Omega^2+\Delta^2}$ with $|+\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,|1\rangle$

$E_- = -\frac{\hbar}{2}\sqrt{\Omega^2+\Delta^2}$ with $|-\rangle = \sin\frac{\theta}{2}\,|0\rangle - \cos\frac{\theta}{2}\,|1\rangle$.

6. $|\psi(0)\rangle = |\tilde{\psi}(0)\rangle = |0\rangle = \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle$

$\Rightarrow \quad |\tilde{\psi}(t)\rangle = \cos\frac{\theta}{2}\,e^{-i\sqrt{\Omega^2+\Delta^2}\,\frac{t}{2}}\,|+\rangle + \sin\frac{\theta}{2}\,e^{i\sqrt{\Omega^2+\Delta^2}\,\frac{t}{2}}\,|-\rangle$

7. $p_1(t) = |\langle 1|\psi(t)\rangle|^2 = |\langle 1|\tilde{\psi}(t)\rangle|^2$

in initial frame — in rotating frame.

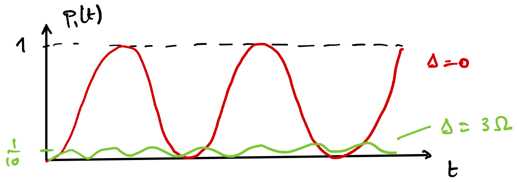
$$= \left| \cos\frac{\theta}{2}\, e^{-i\sqrt{\ }\,\frac{t}{2}} \langle 1|+\rangle + \sin\frac{\theta}{2}\, e^{i\sqrt{\ }\,\frac{t}{2}} \langle 1|-\rangle \right|^2$$

$$= \left| 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} \left[\frac{e^{-i\sqrt{\ }\,\frac{t}{2}} - e^{i\sqrt{\ }\,\frac{t}{2}}}{2} \right] \right|^2$$

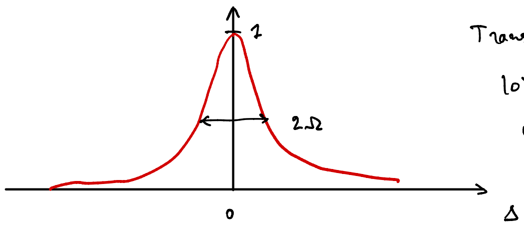
$$= \underbrace{\sin^2\theta}_{\frac{\Omega^2}{\Omega^2+\Delta^2}} \sin^2\left[\sqrt{\Omega^2+\Delta^2}\,\frac{t}{2}\right]$$

which is the Rabi formula.

8.



9. Envelope : $\dfrac{\Omega^2}{\Omega^2+\Delta^2}$ Lorentzian curve.



Transfer from $|0\rangle$ to $|1\rangle$ maximal on resonance ($\Delta = 0$)

III Zeno effect

1. $\Delta = 0 \Rightarrow P_0(\delta t) = \cos^2\left[\frac{\Omega \delta t}{2}\right] = 1 - \frac{\Omega^2 \delta t^2}{4} + O(\Omega^4 \delta t^4)$

2. $P_0(T) = \left[P_0(\delta t)\right]^N \approx \left(1 - \frac{\Omega^2 T^2}{4N^2}\right)^N \simeq e^{-\frac{\Omega^2 T^2}{4N}}$

3. $\lim_{N \to \infty} P_1(T) = 0$

4. If you do not measure every δt, $P_0(T) = \cos^2 \frac{\Omega T}{2}$.

   Measuring the system projects it into $|0\rangle$, and prevents the system from evolving: a continuous measurement freezes the state of the system.

IV Entangling gate

1. $$|\psi_{in}\rangle = \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{c} \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{t}$$

   $$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

   $$|\psi_{out}\rangle = U_\pi^{(2)} |\psi_{in}\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

2. If not entangled, $\exists\ \alpha, \beta, \gamma\ \delta$ such that:

   $$|\psi_{out}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

   $$\Rightarrow \underbrace{\alpha\gamma = \alpha\delta = \beta\gamma = \frac{1}{2}} \quad \text{and} \quad \beta\delta = -\frac{1}{2} \quad \Bigg\} \text{ incompatible}$$

   $$\alpha^2\gamma^2\beta\delta = \frac{1}{8} > 0 \Rightarrow \beta\delta > 0$$

   Hence not separable $\Rightarrow$ entangled.

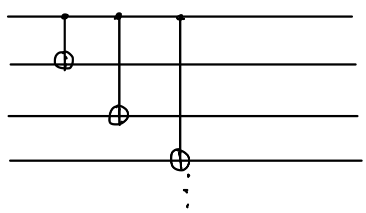
3. $U_\pi^{(2)}$ leave the c qbit untouched, hence one just has to look at:

   $$U_H \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{U_\pi^{(2)} \text{ for the target qbit}} U_H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

   $$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ c

   $N\times$ $|0\rangle$, $|0\rangle$, $|0\rangle$ t



$$\frac{1}{\sqrt{2}}\left(\underbrace{|0 \dots 0\rangle}_{N+1} + \underbrace{|1 \dots 1\rangle}_{N+1}\right)$$