Physics of Quantum Information exam (3 hours)

Master 2 QLMN

1 December 2023 (09:00 - 12:00)

Allowed documents : 4 pages of personal notes. Calculators allowed.
No lecture notes, no HW, no tablet, no smartphone nor any other electronic device.
Aux francophones (et francographes) : Vous pouvez répondre en français.

The exam consists of three independent parts :

1. General independent questions about the course ;
2. A problem on the detection loophole associated to the Bell's inequality.
3. A problem about a model of decoherence.

Many questions are qualitative and do not require long explanations. Answer to them in a concise and precise way !

1 General questions about the course

1. Represent on the Bloch sphere the evolution under $H = (\hbar\Omega/2)\,\sigma_x$ of a qubit initially prepared in the state $(|0\rangle + i\,|1\rangle)/\sqrt{2}$. What happens if the qubit is initially prepared in $(|0\rangle + |1\rangle)/\sqrt{2}$?
2. Take a two-level atom placed in an optical cavity. The coupling between the atom and the cavity mode is ruled by the Hamiltonian $H = \hbar g(a^{+}\sigma^{-} + a\sigma^{+})$. The cavity is resonant with the atom. One of the cavity mirrors is only partially reflecting, so that the energy can leak out from the cavity with a rate κ (expressed in s^{-1}). The atom is initially prepared in its excited state $|e\rangle$ and the cavity is empty.
   (a) Plot the temporal evolution of the probability $P_e(t)$ to find the atom in its excited state $|e\rangle$ in the two cases : $g \gg \kappa$, and $g \ll \kappa$. Show the timescales on the graph.
   (b) In which case, $g \gg \kappa$ or $g \ll \kappa$, can the atom be described as being coupled to a Markovian bath.
   (c) Estimate the correlation time τ_c of the bath in this case as a function of the parameters of the problem.

3. Consider two qubits A and B in the state

$$|\Psi\rangle_{AB} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + e^{i\phi}|11\rangle).$$

   (a) Is this state normalized ?
   (b) For which value of ϕ is this a product state ? When is it an entangled state ?
   (c) Calculate the density operator of A.
   (d) Calculate the von Neumann entropy of A, $S_A(\phi)$.
   (e) Plot S_A as a function of $x = \cos(\phi/2)$ and find the value of ϕ for which the state is maximally entangled.

4. One wants to perform the tomography of the density matrix of two qubits.
   (a) Recall briefly how to do it.
   (b) The experiment can be repeated at a rate of 10 Hz. To obtain enough statistics one repeats each measurement 500 times. Estimate the time required to performed the full state tomography of the two qubits.

2 Bell inequalities and detection loophole

We have seen in the course that a two-photon entangled state violates the Bell inequality. All the experiments performed before 2001 to check this operated with photon detectors with finite efficiencies. One thus had to assume a fair sampling, i.e. that the pairs detected, and violating Bell's inequality, were representative also of the undetected ones : Otherwise, one could argue that the pairs that *are not* detected could yield a contribution such that the ensemble of detected and undetected pairs fulfill the Bell inequality, so that the experiment could be described by a local hidden variable theory. The first experimental demonstration of a Bell inequality closing this "detection loophole" was performed in 2001 by D.J. Wineland *et al.* [Nature, **409**, 791 (2001)] with a pair of trapped Be^+ ions rather than photons. The qubit states $(|0\rangle, |1\rangle)$ was encoded on two hyperfine states of each ion, and the pair was prepared in the Bell state $|\psi\rangle = (|0,1\rangle - |1,0\rangle)/\sqrt{2}$. We study here the influence of the finite efficiency of the detection.

5. To test the Bell inequality, we introduce the correlator $E(\theta_A, \theta_B) = \langle \hat{\sigma}_{\theta_A} \otimes \hat{\sigma}_{\theta_B} \rangle$ with $\hat{\sigma}_{\theta_{A,B}} = \cos(\theta_{A,B})\hat{\sigma}_x + \sin(\theta_{A,B})\hat{\sigma}_y$. Calculate $E(\theta_A, \theta_B)$ as a function of $\theta_A - \theta_B$. The expression of the Pauli matrices in the basis $\{|0\rangle, |1\rangle\}$ are :

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1}$$

   Hint : Note that $\hat{\sigma}_\theta$ has a simple expression in the basis $\{|0\rangle, |1\rangle\}$.

6. To describe the effect of an imperfect detection, we model the outcome of the experiment by the following density operator

$$\hat{\rho} = \eta|\psi\rangle\langle\psi| + (1-\eta)\hat{\rho}' , \tag{2}$$

where η (≤ 1) is the efficiency of the detection and $\hat{\rho}'$ describes a *fully uncorrelated* state (i.e. such that $\langle \hat{A} \otimes \hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle$, for any operators $\hat{A}$ and $\hat{B}$ acting on A and B, respectively). Use the statistical interpretation of the density operator to explain why this is a good way of modeling an imperfect detection.

7. Calculate $E(\theta_A, \theta_B) = \mathrm{Tr}[\hat{\rho}\, \hat{\sigma}_{\theta_\mathrm{A}} \otimes \hat{\sigma}_{\theta_\mathrm{B}}]$ for the density operator (2), as a function of $\theta_A - \theta_B$ and η.
8. The Bell parameter is defined as $S(\theta_A, \theta_B, \theta'_A, \theta'_B) = E(\theta_A, \theta_B) + E(\theta'_A, \theta_B) + E(\theta'_A, \theta'_B) - E(\theta_A, \theta'_B)$. We recall that for a local hidden variable theory, $S_{\mathrm{LHV}} \leq 2$ for any choice of the angles. For the state $|\psi\rangle$, $S_\mathrm{Q} = 2\sqrt{2}$ for a choice of the angles $\theta_B - \theta_A = \theta'_A - \theta_B = \theta'_B - \theta'_A = \pi/4$. Calculate the maximal Bell parameter one can obtain for a detection efficiency η.
9. Show that η has to be larger than a threshold to be able to conclude on the violation of the Bell inequality.
10. On the experiment, the state of each atom was measured by shining a laser resonant with the transition coupling one of the hyperfine states to an optically excited state, and by collecting the emitted fluorescence. An example of histogram of the total fluorescence emitted by the two ions collected on the photodetector is shown in Fig. 1.

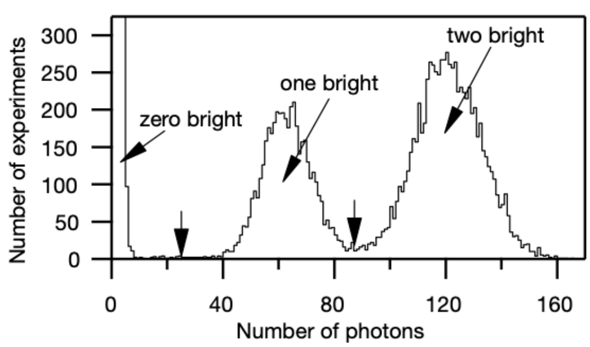

FIGURE 1 – *Histogram of the fluorescence collected from the two ions .*

   (a) Explain how the measurement allows one to distinguish the two qubit states.
   (b) Explain the results shown in Fig. 1. In particular, to which state(s) correspond the peaks "zero bright", "one bright" and "two bright" ?
   (c) Extract the photon counting rate per atom, knowing that the time needed to collect the histogram was 20 sec.
   (d) Based on the histogram, explain why the detection method is of high efficiency.

11. When testing Bell's inequalities with photons, one uses polarizers with different orientations to measure the correlators. In the case of a qubit encoded on ions, how can you measure the state along various axis ?
12. The result of five different runs of the experiment gave the following results for the Bell parameters : $S_{\mathrm{exp}} = 2.222, 2.275, 2.262, 2.246, 2.245$. Estimate by how many standard deviations is the Bell inequality violated.

13. Assuming that the only source of error is the detection error, estimate η.
14. The distance between the two ions was around $5\,\mu$m. Can this experiment be considered as fully conclusive in terms of closing the usual loopholes associated with the violation of Bell's inequalities ?

3 Decoherence of a Schrödinger cat state

The aim of this problem is to study a simple decoherence model in the Kraus formalism and to apply it to a Schrödinger cat state. We consider a qubit, with states $|0\rangle$ and $|1\rangle$, coupled to an unspecified environment. The Pauli matrices, written in the computational basis $\{|0\rangle, |1\rangle\}$, are

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

and that they fulfill the relation $\hat{\sigma}_j^2 = \hat{\mathbb{1}}$, $\forall j \in \{x, y, z\}$.

3.1 Phase flip

Assume first that the qubit dynamics is generated by the two Kraus operators

$$\hat{M}_0 = \sqrt{1-p}\,\hat{\mathbb{1}} \qquad \text{and} \qquad \hat{M}_z = \sqrt{p}\,\hat{\sigma}_z \ , \tag{4}$$

with $p \in [0, 1]$.

15. Check the completeness relation of this set of Kraus operators.
16. Give the physical interpretation of the Kraus operators $\hat{M}_0$ and $\hat{M}_z$. You may consider their actions on a single-qubit state of the form $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Interpret also the quantity p.
17. Write the Kraus map, $\hat{\rho}^{(1)} = \mathcal{K}_z[\hat{\rho}]$, as a function of the initial density operator $\hat{\rho}$ of the qubit.

3.2 Qubit in a superposition state

We first consider a single qubit, initially in a pure state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$.

18. Write the density matrix associated with the state $|\psi\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$,
$$\hat{\rho} = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \ , \tag{5}$$
as a function of α and β.
19. Using the Kraus map $\mathcal{K}_z$ written in question 17, show that the variation of the qubit density matrix, $\Delta\hat{\rho} = \hat{\rho}^{(1)} - \hat{\rho}$, may be written as
$$\frac{\Delta\hat{\rho}}{\Delta t} = \frac{\gamma}{2}\left(\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z - \hat{\rho}\right) \ , \tag{6}$$

where Δt is the application time of the map. Equating the quantity $\Delta\hat{\rho}/\Delta t$ with the derivative with respect to time, $d\hat{\rho}/dt$, justify that this is a Lindblad form. Give the expression of γ as a function of p and Δt, and interpret it.

20. Integrate the Lindblad equation by writing the differential equations for the four components of the density matrix.
21. Explain the effect of the phase-flip process on the qubit state.

3.3 Macroscopic Schrödinger cat

We now consider a N-qubit Schrödinger cat in the form of the GHZ state

$$|\mathrm{GHZ}\rangle = \frac{|00...0\rangle + |11...1\rangle}{\sqrt{2}} \ . \tag{7}$$

22. Assume that the N qubits can undergo independent phase flips. What is the effect of the Kraus operator $\hat{\sigma}_z^{(j)}$ (Pauli matrix along z acting on spin j only) on the state $|\mathrm{GHZ}\rangle$?
23. Conclude that under a phase flip process the density matrix $\hat{\rho}$ remains in the two-dimensional sub-Hilbert generated by the states $|00...0\rangle$ and $|11...1\rangle$.
24. Show that the Lindblad equation may be written in a form similar to Eq. (6),

$$\frac{d\hat{\rho}}{dt} = \frac{\Gamma}{2}\left(\hat{Z}\hat{\rho}\hat{Z} - \hat{\rho}\right) \ . \tag{8}$$

    Give the expression of Γ as a function of γ and N. Write the operator $\hat{Z}$ as a sum of projectors involving $|00...0\rangle$ and $|11...1\rangle$.

25. Conclude about the decoherence of a macroscopic cat state, compared to a single qubit. Use a numerical example to show that a large cat state decoheres very rapidly.